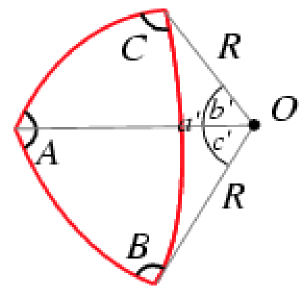
Spherical Trigonometry



Let a spherical triangle be drawn on the surface of a sphere of radius R, centered at a point $O=(0,0,0)$, with vertices A, B, and C. The vectors from the center of the sphere to the vertices are therefore given by $\mathbf{a}\equiv\overrightarrow{OA}$, $\mathbf{b}\equiv\overrightarrow{OB}$, and $\mathbf{c}\equiv\overrightarrow{OC}$. Now, the *angular* lengths of the sides of the triangle (in radians) are then $a'\equiv\angle BOC$, $b'\equiv\angle COA$, and $c'\equiv\angle AOB$, and the *actual* arc lengths of the side are $a=Ra'$, $b=Rb'$, and $c=Rc'$. Explicitly,

$$\mathbf{a}\cdot\mathbf{b} = R^2\cos c' = R^2\cos\left(\frac{c}{R}\right) \quad (1)$$

$$\mathbf{a}\cdot\mathbf{c} = R^2\cos b' = R^2\cos\left(\frac{b}{R}\right) \quad (2)$$

$$\mathbf{b}\cdot\mathbf{c} = R^2\cos a' = R^2\cos\left(\frac{a}{R}\right). \quad (3)$$

Now make use of A, B, and C to denote both the vertices themselves and the *angles* of the spherical triangle at these vertices, so that the dihedral angle between planes AOB and AOC is written A, the dihedral angle between planes BOC and AOB is written B, and the dihedral angle between planes BOC and AOC is written C. (These angles are sometimes instead denoted α, β, γ; e.g., Gellert *et al.* 1989)

Consider the dihedral angle A between planes AOB and AOC, which can be calculated using the dot product of the normals to the planes. Assuming $R=1$, the normals are given by cross products of the vectors to the vertices, so

$$(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{c}}) = (|\hat{\mathbf{a}}|\,|\hat{\mathbf{b}}|\sin c)(|\hat{\mathbf{a}}|\,|\hat{\mathbf{c}}|\sin b)\cos A \quad (4)$$

$$= \sin b\sin c\cos A. \quad (5)$$

However, using a well-known vector identity gives

$$\begin{aligned}(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{c}}) &= \hat{\mathbf{a}}\cdot\left[\hat{\mathbf{b}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{c}})\right] & (6)\\ &= \hat{\mathbf{a}}\cdot\left[\hat{\mathbf{a}}\left(\hat{\mathbf{b}}\cdot\hat{\mathbf{c}}\right)-\hat{\mathbf{c}}\left(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}\right)\right] & (7)\\ &= \left(\hat{\mathbf{b}}\cdot\hat{\mathbf{c}}\right)-(\hat{\mathbf{a}}\cdot\hat{\mathbf{c}})\left(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}\right) & (8)\\ &= \cos a-\cos c\cos b. & (9)\end{aligned}$$

Since these two expressions must be equal, we obtain the identity (and its two analogous formulas)

$$\begin{aligned}\cos a &= \cos b\cos c+\sin b\sin c\cos A & (10)\\ \cos b &= \cos c\cos a+\sin c\sin a\cos B & (11)\\ \cos c &= \cos a\cos b+\sin a\sin b\cos C, & (12)\end{aligned}$$

known as the cosine rules for sides (Smart 1960, pp. 7-8; Gellert *et al.* 1989, p. 264; Zwillinger 1995, p. 469).

The identity

$$\begin{aligned}\sin A &= \frac{\left|\left(\hat{\mathbf{a}}\times\hat{\mathbf{b}}\right)\times(\hat{\mathbf{a}}\times\hat{\mathbf{c}})\right|}{\left|\hat{\mathbf{a}}\times\hat{\mathbf{b}}\right|\,|\hat{\mathbf{a}}\times\hat{\mathbf{c}}|} & (13)\\ &= -\frac{\left|\hat{\mathbf{a}}\left[\hat{\mathbf{b}},\,\hat{\mathbf{a}},\,\hat{\mathbf{c}}\right]+\hat{\mathbf{b}}\,[\hat{\mathbf{a}},\,\hat{\mathbf{a}},\,\hat{\mathbf{c}}]\right|}{\sin b\sin c} & (14)\\ &= \frac{\left[\hat{\mathbf{a}},\,\hat{\mathbf{b}},\,\hat{\mathbf{c}}\right]}{\sin b\sin c}, & (15)\end{aligned}$$

where $[\mathbf{a},\,\mathbf{b},\,\mathbf{c}]$ is the scalar triple product, gives

$$\frac{\sin A}{\sin a}=\frac{\left[\hat{\mathbf{a}},\,\hat{\mathbf{b}},\,\hat{\mathbf{c}}\right]}{\sin a\sin b\sin c}, \qquad (16)$$

so the spherical analog of the law of sines can be written

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}=\frac{6\,\mathrm{Vol}\,(O\,A\,B\,C)}{\sin a\sin b\sin c} \qquad (17)$$

(Smart 1960, pp. 9-10; Gellert *et al.* 1989, p. 265; Zwillinger 1995, p. 469), where $\mathrm{Vol}\,(O\,A\,B\,C)$ is the volume of the tetrahedron.

The analogs of the law of cosines for the angles of a spherical triangle are given by

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \quad (18)$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b \quad (19)$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \quad (20)$$

(Gellert *et al.* 1989, p. 265; Zwillinger 1995, p. 470).

Finally, there are spherical analogs of the law of tangents,

$$\frac{\tan\left[\frac{1}{2}(B-C)\right]}{\tan\left[\frac{1}{2}(B+C)\right]} = \frac{\tan\left[\frac{1}{2}(b-c)\right]}{\tan\left[\frac{1}{2}(b+c)\right]} \quad (21)$$

$$\frac{\tan\left[\frac{1}{2}(C-A)\right]}{\tan\left[\frac{1}{2}(C+A)\right]} = \frac{\tan\left[\frac{1}{2}(c-a)\right]}{\tan\left[\frac{1}{2}(c+a)\right]} \quad (22)$$

$$\frac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]} = \frac{\tan\left[\frac{1}{2}(a-b)\right]}{\tan\left[\frac{1}{2}(a+b)\right]} \quad (23)$$

(Beyer 1987; Gellert *et al.* 1989; Zwillinger 1995, p. 470).

Additional important identities are given by

$$\cos A = \csc b \csc c (\cos a - \cos b \cos c), \quad (24)$$

(Smart 1960, p. 8),

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A \quad (25)$$

(Smart 1960, p. 10), and

$$\cos a \cos C = \sin a \cot b - \sin C \cot B \quad (26)$$

(Smart 1960, p. 12).

Let

$$s \equiv \frac{1}{2}(a+b+c) \quad (27)$$

be the semiperimeter, then half-angle formulas for sines can be written as

$$\sin\left(\tfrac{1}{2}A\right) = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}} \quad (28)$$

$$\sin\left(\tfrac{1}{2}B\right) = \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}} \quad (29)$$

$$\sin\left(\tfrac{1}{2}C\right) = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}, \quad (30)$$

for cosines can be written as

$$\cos\left(\tfrac{1}{2}A\right) = \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}} \quad (31)$$

$$\cos\left(\tfrac{1}{2}B\right) = \sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}} \quad (32)$$

$$\cos\left(\tfrac{1}{2}C\right) = \sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}, \quad (33)$$

and tangents can be written as

$$\tan\left(\tfrac{1}{2}A\right) = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}} = \frac{k}{\sin(s-a)} \quad (34)$$

$$\tan\left(\tfrac{1}{2}B\right) = \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}} = \frac{k}{\sin(s-b)} \quad (35)$$

$$\tan\left(\tfrac{1}{2}C\right) = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}} = \frac{k}{\sin(s-c)}, \quad (36)$$

where

$$k^2 = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s} \quad (37)$$

(Smart 1960, pp. 8-9; Gellert *et al.* 1989, p. 265; Zwillinger 1995, p. 470).

Let

$$S \equiv \tfrac{1}{2}(A+B+C) \quad (38)$$

be the sum of half-angles, then the half-side formulas are

$$\tan\left(\tfrac{1}{2}a\right) = K\cos(S-A) \quad (39)$$
$$\tan\left(\tfrac{1}{2}b\right) = K\cos(S-B) \quad (40)$$
$$\tan\left(\tfrac{1}{2}c\right) = K\cos(S-C), \quad (41)$$

where

$$K^2 = -\frac{\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)} \quad (42)$$

(Gellert *et al.* 1989, p. 265; Zwillinger 1995, p. 470).

The haversine formula for sides, where

$$\operatorname{hav} x \equiv \tfrac{1}{2}(1-\cos x) = \sin^2\left(\tfrac{1}{2}x\right), \quad (43)$$

is given by

$$\operatorname{hav} a = \operatorname{hav}(b-c) + \sin b\sin c\operatorname{hav} A \quad (44)$$

(Smart 1960, pp. 18-19; Zwillinger 1995, p. 471), and the haversine formula for angles is given by

$$\operatorname{hav} A = \frac{\sin(s-b)\sin(s-c)}{\sin b\sin c} \quad (45)$$
$$= \frac{\operatorname{hav} a - \operatorname{hav}(b-c)}{\sin b\sin c} \quad (46)$$
$$= \operatorname{hav}[\pi-(B+C)] + \sin B\sin C\operatorname{hav} a \quad (47)$$

(Zwillinger 1995, p. 471).

Gauss's formulas (also called Delambre's analogies) are

$$\frac{\sin\left[\tfrac{1}{2}(a-b)\right]}{\sin\left(\tfrac{1}{2}c\right)} = \frac{\sin\left[\tfrac{1}{2}(A-B)\right]}{\cos\left(\tfrac{1}{2}C\right)} \quad (48)$$
$$\frac{\sin\left[\tfrac{1}{2}(a+b)\right]}{\sin\left(\tfrac{1}{2}c\right)} = \frac{\cos\left[\tfrac{1}{2}(A-B)\right]}{\sin\left(\tfrac{1}{2}C\right)} \quad (49)$$
$$\frac{\cos\left[\tfrac{1}{2}(a-b)\right]}{\cos\left(\tfrac{1}{2}c\right)} = \frac{\sin\left[\tfrac{1}{2}(A+B)\right]}{\cos\left(\tfrac{1}{2}C\right)} \quad (50)$$

$$\frac{\cos\left[\frac{1}{2}(a+b)\right]}{\cos\left(\frac{1}{2}c\right)} = \frac{\cos\left[\frac{1}{2}(A+B)\right]}{\sin\left(\frac{1}{2}C\right)} \quad (51)$$

(Smart 1960, p. 22; Zwillinger 1995, p. 470).

Napier's analogies are

$$\frac{\sin\left[\frac{1}{2}(A-B)\right]}{\sin\left[\frac{1}{2}(A+B)\right]} = \frac{\tan\left[\frac{1}{2}(a-b)\right]}{\tan\left(\frac{1}{2}c\right)} \quad (52)$$

$$\frac{\cos\left[\frac{1}{2}(A-B)\right]}{\cos\left[\frac{1}{2}(A+B)\right]} = \frac{\tan\left[\frac{1}{2}(a+b)\right]}{\tan\left(\frac{1}{2}c\right)} \quad (53)$$

$$\frac{\sin\left[\frac{1}{2}(a-b)\right]}{\sin\left[\frac{1}{2}(a+b)\right]} = \frac{\tan\left[\frac{1}{2}(A-B)\right]}{\cot\left(\frac{1}{2}C\right)} \quad (54)$$

$$\frac{\cos\left[\frac{1}{2}(a-b)\right]}{\cos\left[\frac{1}{2}(a+b)\right]} = \frac{\tan\left[\frac{1}{2}(A+B)\right]}{\cot\left(\frac{1}{2}C\right)} \quad (55)$$

(Beyer 1987; Gellert *et al.* 1989, p. 266; Zwillinger 1995, p. 471).

SEE ALSO: Angular Defect, Descartes Total Angular Defect, Gauss's Formulas, Girard's Spherical Excess Formula, Law of Cosines, Law of Sines, Law of Tangents, L'Huilier's Theorem, Napier's Analogies, Solid Angle, Spherical Excess, Spherical Geometry, Spherical Polygon, Spherical Triangle. [Pages Linking Here]

REFERENCES:

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 131 and 147-150, 1987.

Danby, J. M. *Fundamentals of Celestial Mechanics, 2nd ed., rev. ed.* Richmond, VA: Willmann-Bell, 1988.

Gellert, W.; Gottwald, S.; Hellwich, M.; Kästner, H.; and Künstner, H. (Eds.). "Spherical Trigonometry." §12 in *VNR Concise Encyclopedia of Mathematics, 2nd ed.* New York: Van Nostrand Reinhold, pp. 261-282, 1989.

Green, R. M. *Spherical Astronomy.* New York: Cambridge University Press, 1985.

Smart, W. M. *Text-Book on Spherical Astronomy, 6th ed.* Cambridge, England: Cambridge University Press, 1960.

Zwillinger, D. (Ed.). "Spherical Geometry and Trigonometry." §6.4 in *CRC Standard Mathematical Tables and Formulae.* Boca Raton, FL: CRC

Press, pp. 468-471, 1995.

CITE THIS AS:

Eric W. Weisstein. "Spherical Trigonometry." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/SphericalTrigonometry.html