Complex Number

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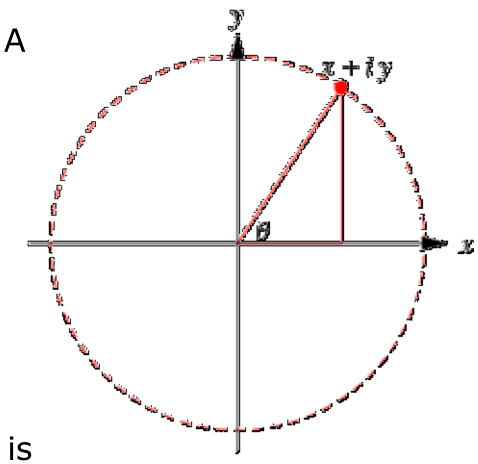
The complex numbers are the field $\mathbb{C}$ of numbers of the form $x + i y$, where x and y are real numbers and i is the imaginary unit equal to the square root of -1, $\sqrt{-1}$. When a single letter $z = x + i y$ is used to denote a complex number, it is sometimes called an "affix." In component notation, $z = x + i y$ can be written (x, y). The field of complex numbers includes the field of real numbers as a subfield.

The set of complex numbers is implemented in *Mathematica* as Complexes. A number x can then be tested to see if it is complex using the command Element[x, Complexes], and expressions that are complex numbers have the Head of Complex.



Complex numbers are useful abstract quantities that can be used in calculations and result in physically meaningful solutions. However, recognition of this fact is one that took a long time for mathematicians to accept. For example, John Wallis wrote, "These Imaginary Quantities (as they are commonly called) arising from the Supposed Root of a Negative Square (when they happen) are reputed to imply that the Case proposed is Impossible" (Wells 1986, p. 22).

Through the Euler formula, a complex number

$$z = x + i y \tag{1}$$

may be written in "phasor" form

$$z = |z| (\cos\theta + i \sin\theta) = |z| e^{i\theta}. \tag{2}$$

Here, $|z|$ is known as the complex modulus (or sometimes the complex norm) and θ is known as the complex argument or phase. The plot above shows what is known as an Argand diagram of the point z , where the dashed circle represents the complex modulus $|z|$ of z and the angle θ represents its complex argument. Historically, the geometric representation of a complex number as simply a point in the plane was important because it made the whole idea of a complex number more acceptable. In particular, "imaginary" numbers became accepted partly through their visualization.

Unlike real numbers, complex numbers do not have a natural ordering, so there is no analog of complex-valued inequalities. This property is not so surprising however when they are viewed as being elements in the complex plane, since points in a plane also lack a natural ordering.

The absolute square of z is defined by $|z|^2 = z\,\bar{z}$, with $\bar{z}$ the complex conjugate, and the argument may be computed from

$$\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{3}$$

The real $\Re(z)$ and imaginary parts $\Im(z)$ are given by

$$\Re(z) = \frac{1}{2}(z + \bar{z}) \tag{4}$$

$$\Im(z) = \frac{z - \bar{z}}{2i} = -\frac{1}{2}i(z - \bar{z}) = \frac{1}{2}i(\bar{z} - z). \tag{5}$$

de Moivre's identity relates powers of complex numbers for real n by

$$z^n = |z|^n\,[\cos(n\theta) + i\sin(n\theta)]. \tag{6}$$

A power of complex number z to a positive integer exponent n can be written in closed form as

$$z^n = \left[x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 - \ldots\right] + i\left[\binom{n}{1} x^{n-1} y - \binom{n}{3} x^{n-3} y^3 + \ldots\right]. \tag{7}$$

The first few are explicitly

$$z^2 = (x^2 - y^2) + i\,(2\,x\,y) \tag{8}$$
$$z^3 = (x^3 - 3\,x\,y^2) + i\,(3\,x^2\,y - y^3) \tag{9}$$
$$z^4 = (x^4 - 6\,x^2\,y^2 + y^4) + i\,(4\,x^3\,y - 4\,x\,y^3) \tag{10}$$
$$z^5 = (x^5 - 10\,x^3\,y^2 + 5\,x\,y^4) + i\,(5\,x^4\,y - 10\,x^2\,y^3 + y^5) \tag{11}$$

(Abramowitz and Stegun 1972).

Complex addition

$$(a + b\,i) + (c + d\,i) = (a + c) + i\,(b + d), \tag{12}$$

complex subtraction

$$(a + b\,i) - (c + d\,i) = (a - c) + i\,(b - d), \tag{13}$$

complex multiplication

$$(a + b\,i)\,(c + d\,i) = (a\,c - b\,d) + i\,(a\,d + b\,c), \tag{14}$$

and complex division

$$\frac{a+b\,i}{c+d\,i} = \frac{(a\,c+b\,d)+i\,(b\,c-a\,d)}{c^2+d^2} \tag{15}$$

can also be defined for complex numbers.

Complex numbers may also be taken to complex powers. For example, complex exponentiation obeys

$$(a+b\,i)^{c+d\,i} = (a^2+b^2)^{(c+i\,d)/2}\,e^{i\,(c+i\,d)\arg(a+i\,b)}, \tag{16}$$

where $\arg(z)$ is the complex argument.

SEE ALSO: Absolute Square, Argand Diagram, Complex Argument, Complex Division, Complex Exponentiation, Complex Modulus, Complex Multiplication, Complex Plane, Complex Subtraction, *i*, Imaginary Number, Phase, Phasor, Real Number, Surreal Number.

Euler Formula

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The Euler formula, sometimes also called the euler identity (Trott 2004, p. 174), states

$$e^{ix} = \cos x + i \sin x, \quad (1)$$

where i is the imaginary unit. Note that Euler's polyhedral formula is sometimes also called the Euler formula, as is the Euler curvature formula. The equivalent expression

$$i x = \ln(\cos x + i \sin x) \quad (2)$$

had previously been published by Cotes (1714).

The special case of the formula with $x = \pi$ gives the beautiful identity

$$e^{i\pi} + 1 = 0, \quad (3)$$

an equation connecting the fundamental numbers i, pi, e, 1, and 0 (zero), the fundamental operations $+$, $\times$, and exponentiation, the most important relation $=$, and nothing else. Gauss is reported to have commented that if this formula was not immediately obvious, the reader would never be a first-class mathematician (Derbyshire 2004, p. 202).

The Euler formula can be demonstrated using a series expansion

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \quad (4)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} \quad (5)$$

$$= \cos x + i \sin x. \quad (6)$$

It can also be demonstrated using a complex integral. Let

$$z \equiv \cos\theta + i \sin\theta \quad (7)$$

$$dz = (-\sin\theta + i \cos\theta)\, d\theta \quad (8)$$

$$= i(\cos\theta + i \sin\theta)\, d\theta \quad (9)$$

$$= i z\, d\theta \quad (10)$$

$$\int \frac{dz}{z} = \int i\, d\theta \quad (11)$$

$$\ln z = i\theta, \quad (12)$$

so

$$z = e^{i\theta} \quad (13)$$

$$\equiv \cos\theta + i \sin\theta. \quad (14)$$

SEE ALSO: de Moivre's Identity, Euler Identity, Polyhedral Formula. [Pages Linking Here]

Complex Exponentiation

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A complex number may be taken to the power of another complex number. In particular, complex exponentiation satisfies

$$(a+b\,i)^{c+d\,i} = (a^2+b^2)^{(c+i\,d)/2}\,e^{i\,(c+i\,d)\arg(a+i\,b)}, \tag{1}$$

where $\arg(z)$ is the complex argument. Written explicitly in terms of real and imaginary parts,

$$\begin{aligned}(a+b\,i)^{c+d\,i} &= (a^2+b^2)^{c/2}\,e^{-d\arg(a+i\,b)} \\ &\quad \{\cos[c\arg(a+i\,b)+\tfrac{1}{2}\,d\ln(a^2+b^2)]+i\sin[c\arg(a+i\,b)+\tfrac{1}{2}\,d\ln(a^2+b^2)]\}.\end{aligned} \tag{2}$$

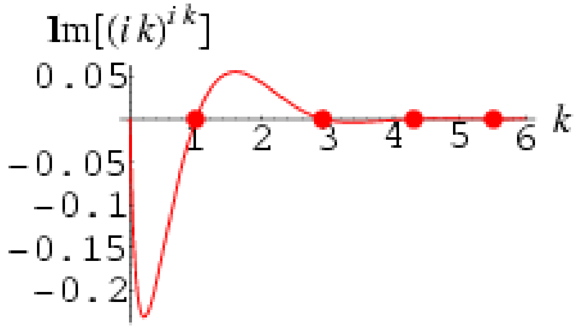
An explicit example of complex exponentiation is given by

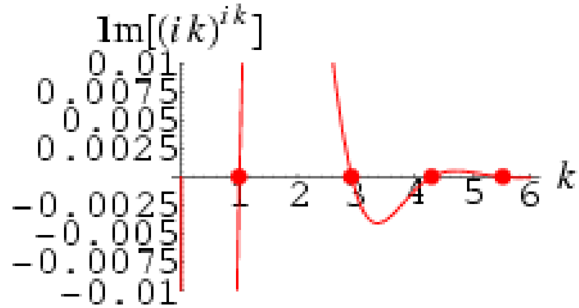
$$(1+i)^{1+i} = \sqrt{2}\,e^{-\pi/4}\,[\cos(\tfrac{1}{4}\,\pi+\tfrac{1}{2}\ln 2)+i\sin(\tfrac{1}{4}\,\pi+\tfrac{1}{2}\ln 2)]. \tag{3}$$

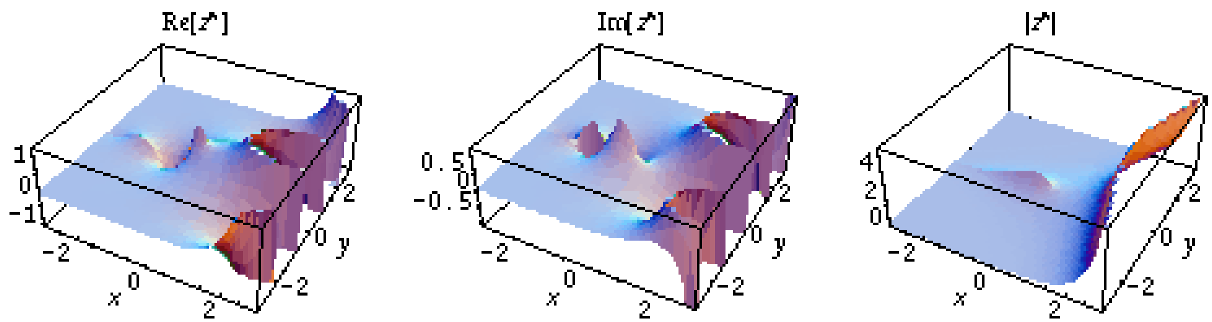
A complex number taken to a complex number can be real. In fact, the famous example

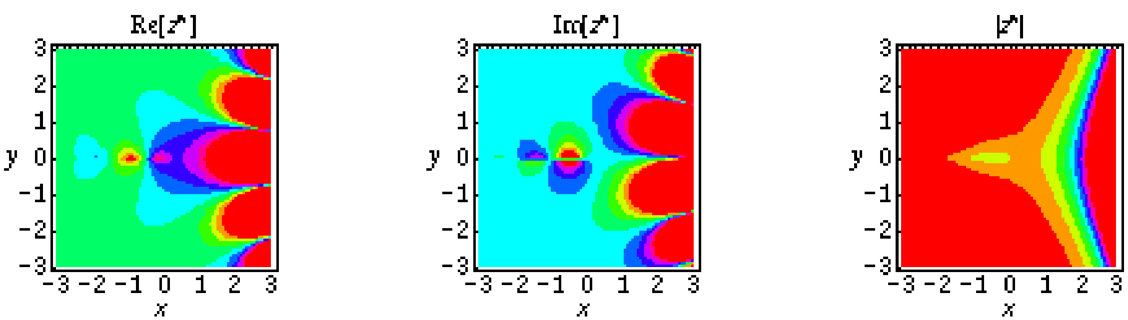
$$i^i = e^{-\pi/2} \tag{4}$$

shows that the power of the purely imaginary i to itself is real.


Im[(i k)^{i k}]
0.05
-0.05
-0.1
-0.15
-0.2
1 2 3 4 5 6
k


Im[(i k)^{i k}]
0.01
0.0075
0.005
0.0025
-0.0025
-0.005
-0.0075
-0.01
1 2 3 4 5 6
k


Re[z^z]
Im[z^z]
|z^z|
x
y


Re[z^z]
Im[z^z]
|z^z|
x
y

In fact, there is a family of values k such that $(i k)^{i k}$ is real, as can be seen by writing

$$(i k)^{i k} = e^{-k \pi/2} [\cos (k \ln k) + i \sin (k \ln k)]. \quad (5)$$

This will be real when $\sin (k \ln k) = 0$, i.e., for

$$k \ln k = n \pi \quad (6)$$

for n an integer. For positive n, this gives roots k_n or

$$k_n = e^{W (n \pi)}, \quad (7)$$

where $W (z)$ is the Lambert W-function. For $n > 1$, this simplifies to

$$k_n = \frac{n \pi}{W (n \pi)}. \quad (8)$$

For $n = 1$, 2, ..., these give the numeric values 1, 2.92606 (Sloane's A088928), 4.30453, 5.51798, 6.63865, 7.6969, ...

SEE ALSO: Complex Addition, Complex Division, Complex Multiplication, Complex Number, Complex Subtraction, Power.
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