Complex Exponentiation

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Let's use z and w to confirm our work as we go along.

$z := 5 - 3i$ $\quad a := 5$ $\quad b := -3$

$w := 2 - 4i$ $\quad c := 2$ $\quad d := -4$

Given: $z^w = (a + b \cdot i)^{c + d \cdot i} = (a + b \cdot i)^c \cdot (a + b \cdot i)^{d \cdot i} = z^c \cdot z^{d \cdot i}$ (1st law of exponents)

Breaking $z^c \cdot z^{d \cdot i}$ **into two parts:** z^c and $z^{d \cdot i}$

Solving z^c using de Moivre's identity:

$$z^c = (a + b \cdot i)^c = (|z|)^c \cdot (\cos(c \cdot \theta) + i \cdot \sin(c \cdot \theta))$$

(where c is a real number)

$$(|z|)^c \cdot (\cos(c \cdot \theta) + i \cdot \sin(c \cdot \theta)) = \left(\sqrt{a^2 + b^2}\right)^c \cdot (\cos(c \cdot \arg(z)) + i \cdot \sin(c \cdot \arg(z)))$$

remembering: $c = 2$ $\quad \arg(z) = -0.54$ $\quad \sqrt{a^2 + b^2} = 5.831$

substituting: $\left(\sqrt{a^2 + b^2}\right)^c \cdot (\cos(c \cdot \arg(z)) + i \cdot \sin(c \cdot \arg(z))) = 16 - 30i$

so. $z^c = 16 - 30i$

Solving z^{di} using Euler's formula: (twice)

write z in polar form $z^{d \cdot i} = \left[\sqrt{a^2 + b^2} \cdot (\cos(\arg(z)) + i \cdot \sin(\arg(z)))\right]^{d \cdot i}$

$e^{ix} = \cos(x) + i\sin(x)$

Euler's formula:

$$\left[\sqrt{a^2 + b^2} \cdot (\cos(\arg(z)) + i \cdot \sin(\arg(z)))\right]^{d \cdot i} = \left(|z| \cdot e^{i \cdot \arg(z)}\right)^{d \cdot i}$$

do you understand this!

$$\left(|z|\cdot e^{i\cdot \arg(z)}\right)^{d\cdot i} = (|z|)^{d\cdot i}\cdot\left(e^{i\cdot \arg(z)}\right)^{d\cdot j}$$ (3rd law of exponents)

simplifying each of these two parts individually:

part 1 $\left(e^{i\cdot \arg(z)}\right)^{d\cdot i} = e^{-1\cdot d\cdot \arg(z)}$ (3rd law of exponents) $e^{-1\cdot d\cdot \arg(z)} = 0.115$ (for this problem)

part 2 $(|z|)^{d\cdot i} = (|z|)^{d\cdot j}$ what to do now?

From http://en.wikipedia.org/wiki/Exponentiation#Summary,

where: $|z| = 1$

$$r^{id} = \left[(r)^d\right]^i = \left[\left(e^{\ln r}\right)^d\right]^i = e^{id\ln r} = \cos(d\ln r) + i\sin(d\ln r).$$

thinking about it and applying it to our $|z|^{di}$ problem yields ----->

$$(|z|)^{d\cdot i} = \left[(|z|)^d\right]^i = e^{\ln\left[(|z|)^{d\cdot i}\right]} = e^{i\cdot d\cdot \ln(|z|)}$$

do you understand this from *logarithms*!

using Euler's formula (*again*):

$$e^{i\cdot d\cdot \ln(|z|)} = \cos(d\cdot \ln(|z|)) + i\cdot \sin(d\cdot \ln(|z|))$$

$$z^{d\cdot i} = (\cos(d\cdot \ln(|z|)) + i\cdot \sin(d\cdot \ln(|z|)))\cdot e^{-1\cdot d\cdot \arg(z)}$$

$$z^{d\cdot i} = 0.083 - 0.08i$$

So putting it all together and checking it with the z and w we've been using:

$$z^{w} = z^{c} \cdot z^{d \cdot i} = \left(\sqrt{a^2 + b^2}\right)^{c} \cdot (\cos(c \cdot \arg(z)) + i \cdot \sin(c \cdot \arg(z))) \cdot \left[\left(e^{-1 \cdot d \cdot \arg(z)}\right) \cdot (\cos(d \cdot \ln(|z|)) + i \cdot \sin(d \cdot \ln(|z|)))\right]$$

$z^{c} \cdot z^{d \cdot i} = (16 - 30i) \cdot (0.083 - 0.08i) = -1.072 - 3.77$

since: $\left(\sqrt{a^2 + b^2}\right)^{c} \cdot (\cos(c \cdot \arg(z)) + i \cdot \sin(c \cdot \arg(z))) = 16 - 30i$

(approximate answer) $\left(e^{-1 \cdot d \cdot \arg(z)}\right) \cdot (\cos(d \cdot \ln(|z|)) + i \cdot \sin(d \cdot \ln(|z|))) = 0.083 - 0.08i$

Just for kicks, let's have the machine confirm our solution using the formula in literature:

Remembering:

$z := 5 - 3$ $a := 5$ $b := -3$ $\arg(z) = -0.54$ $\sqrt{a^2 + b^2} = 5.831$

$w := 2 - 4i$ $c := 2$ $d := -4$ $e = 2.718$ $\ln\left[\sqrt{\left(a^2 + b\right)^2}\right] = 3.091$

$$\text{Answer} := \left(\sqrt{a^2 + b^2}\right)^{c} \cdot (\cos(c \cdot \arg(z)) + i \cdot \sin(c \cdot \arg(z))) \cdot \left[\left(e^{-1 \cdot d \cdot \arg(z)}\right) \cdot (\cos(d \cdot \ln(|z|)) + i \cdot \sin(d \cdot \ln(|z|)))\right]$$

substituting yields our answer: $\text{Answer} = -1.08 - 3.76i$

which is the same as the machine when you computes everything internally: $z^{w} = -1.08 - 3.76i$