The Piston Problem

1. Problem Statement: *Determine the acceleration equation for the engine piston rod in Figure 1 as a function of crankshaft angle and crankshaft RPM. Use the acceleration equation to determine connecting rod stress at a function of RPM and crank angle (θ). Ignore connecting rod mass. Determine at the crank angle which will cause the rod to fail at 10,500 RPM.*

2. Solution technique:

Using calculus, derive the following equations as a function of θ:

a) the piston displacement equation
b) the piston velocity equation
c) the piston acceleration equation

Compute the force and stress equations at the minimum piston rod cross sectional area.

Compute the failure θ at RPM = 10,000.

Let θ be measured from Top Dead Center (TDC) with θ measured positively in a counter clockwise direction.

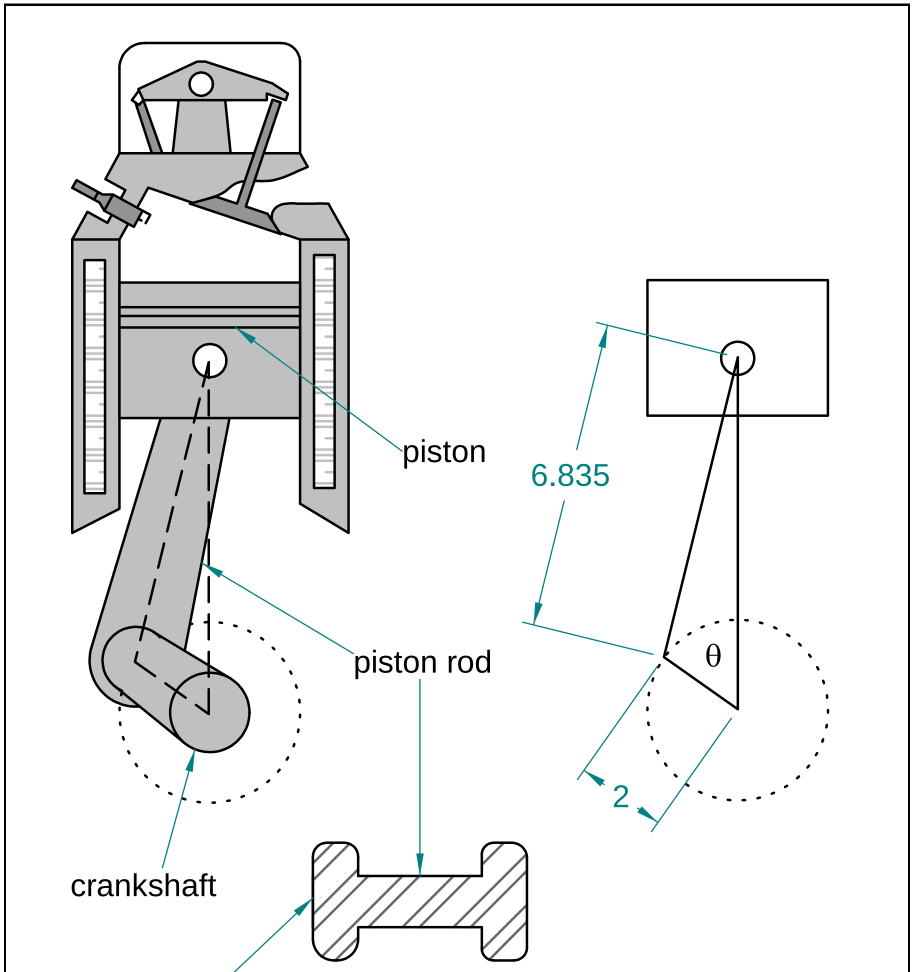
Known values:

$L := 6.835 \cdot in$ $\quad r := 2 \cdot in$

$Mass_{piston} := 3 \cdot lb$ (with pin)

$cross_{sec} := 0.51 \cdot in^2$

Figure 1



Derive the piston displacement equation the the vertical direction as a function of crank angle (θ).

We know from Figure 2 the following relationships (where we let, for example): $\theta := 30\cdot deg$

$$\sin(\theta) = \frac{c}{r} \qquad c := \sin(\theta)\cdot r \qquad \cos(\theta) = \frac{b}{r} \qquad b := \cos(\theta)\cdot r$$

$$L^2 = c^2 + a^2 \qquad a := \left(L^2 - c^2\right)^{\frac{1}{2}}$$ (positive root only)

$$h := b + a \qquad h = 8\,in$$

Changing the equations derived above into functions yields:

$$c(\theta) := \sin(\theta)\cdot r \qquad b(\theta) := \cos(\theta)\cdot r \qquad a(\theta) := \left(L^2 - c(\theta)^2\right)^{\frac{1}{2}}$$

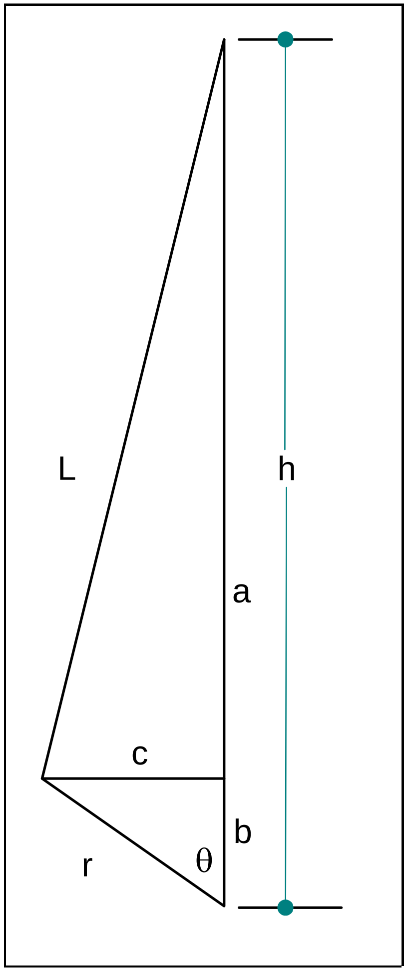
and $h(\theta) := (b(\theta) + a(\theta))$

The displacement equation (as a function of the crank angle (θ)) is therefore:

$$h(\theta) = b(\theta) + a(\theta) = \cos(\theta)\cdot r + \left(L^2 - c(\theta)^2\right)^{\frac{1}{2}} = \cos(\theta)\cdot r + \left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{1}{2}}$$

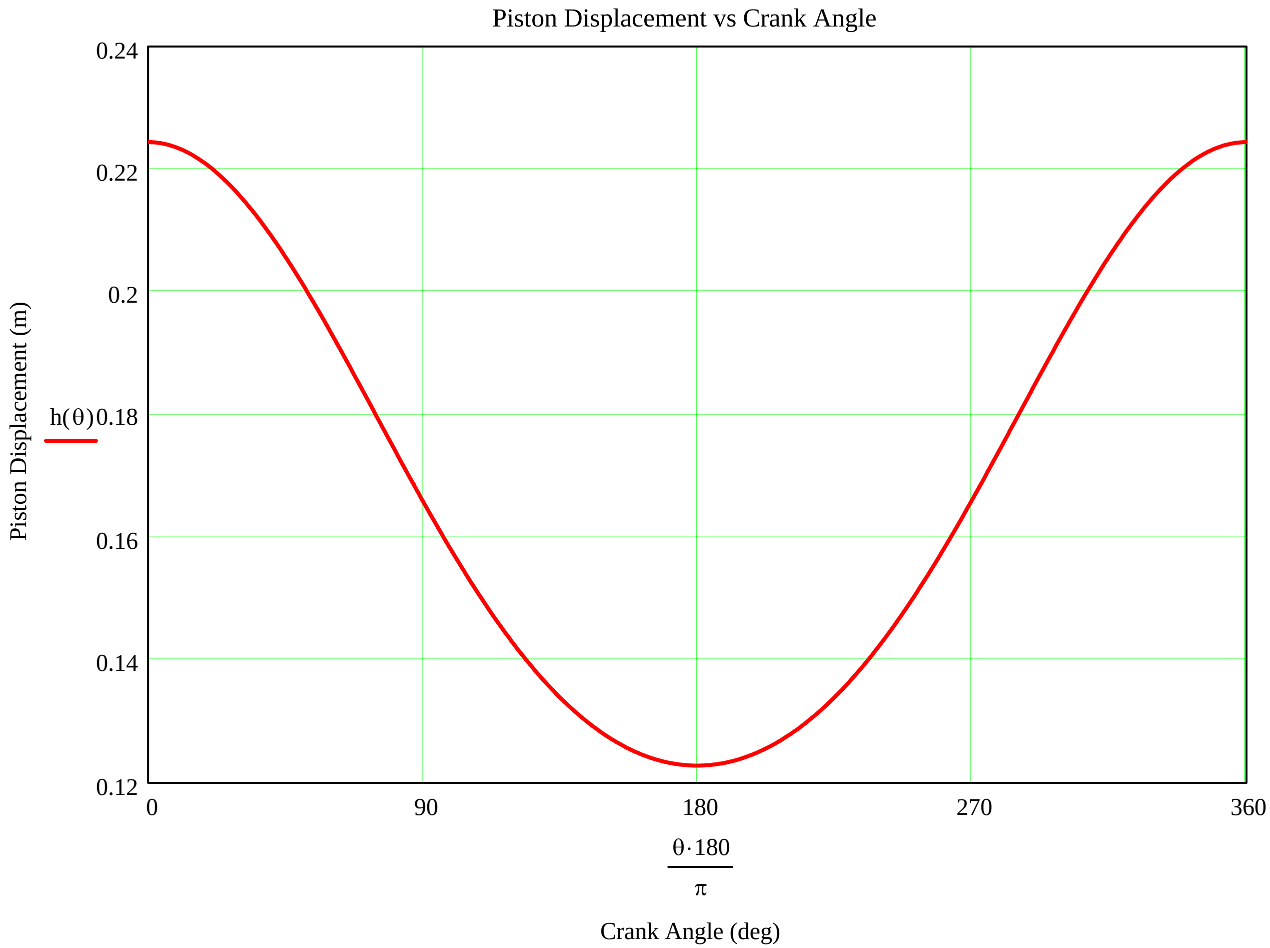
or: $$h(\theta) = \cos(\theta)\cdot r + \sqrt{L^2 - r^2\cdot\sin(\theta)^2}$$

Figure 2



$\theta := 0, .01 .. 2\cdot\pi$

The piston displacement (the linear distance measured from center of crankshaft to piston pin) graph relative to the crank angle in the domain of 0 to 360 degrees is shown to the right.



Calculating piston velocity by taking derivative of the displacement function with respect to time:

$$\frac{d}{dt}h(\theta) = \frac{d}{dt}\left[\cos(\theta)\cdot r + \left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{1}{2}}\right] = \frac{d}{dt}(\cos(\theta)\cdot r) + \frac{d}{dt}\sqrt{L^2 - r^2\cdot\sin(\theta)^2}$$

breaking into 2 parts and recalling ω = angular velocity (RPM) since: $\omega = \frac{d}{dt}\theta$ (the time rate of change of the crank angle)

Part 1 $$\frac{d}{dt}(\cos(\theta)\cdot r) = \cos(\theta)\cdot\frac{d}{dt}r + r\cdot\frac{d}{dt}\cos(\theta) = \cos(\theta)\cdot 0 + r\cdot\left[-\left[\left(\frac{d}{dt}\theta\right)\cdot\sin(\theta)\right]\right] = -r\cdot\omega\cdot\sin(\theta)$$

Part 2 Remembering from calculus: $\frac{d}{dt}\sqrt{u} = \frac{1}{2\cdot\sqrt{u}}\cdot\frac{d}{dt}u$

$$\frac{d}{dt}\sqrt{L^2 - r^2\cdot\sin(\theta)^2} = \frac{1}{2\cdot\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}\cdot\frac{d}{dt}\left(L^2 - r^2\cdot\sin(\theta)^2\right) = \frac{1}{2\cdot\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}\cdot\left[\frac{d}{dt}L^2 - \frac{d}{dt}\left(r^2\cdot\sin(\theta)^2\right)\right]$$

or $$\frac{1}{2\cdot\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}\cdot\left[0 - \left[\left(r^2\right)\cdot\frac{d}{dt}\sin(\theta)^2 + \sin(\theta)^2\cdot\frac{d}{dt}r^2\right]\right] = \frac{1}{2\cdot\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}\cdot\left[0 - \left[\left[r^2\cdot\left(\frac{d}{dt}\sin(\theta)^2\right) + 0\right]\right]\right]$$

remembering:

$$\frac{d}{dt}L^2 = \frac{d}{dt}r^2 = 0$$

since L and r are unchanging.

or $$\frac{1}{2\cdot\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}\cdot\left[-\left(r^2\right)\cdot\left[(\sin(\theta))\cdot\frac{d}{dt}\sin(\theta) + \sin(\theta)\cdot\frac{d}{dt}\sin(\theta)\right]\right] = \frac{1}{2\cdot\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}\cdot\left[-2\cdot\left(r^2\right)\cdot\left[\sin(\theta)\cdot\left(\frac{d}{dt}\sin(\theta)\right)\right]\right]$$

or $$\frac{1}{2\cdot\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}\cdot\left[-2\cdot\left(r^2\right)\cdot\left[\sin(\theta)\cdot\left[\left(\frac{d}{dt}\theta\right)\cdot\cos(\theta)\right]\right]\right] = \frac{1}{2\cdot\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}\cdot\left[-2\cdot\left(r^2\right)\cdot\left[\sin(\theta)\cdot\left[(\omega)\cdot\cos(\theta)\right]\right]\right]$$

or $$\frac{-2\cdot r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)}{2\cdot\sqrt{L^2 - r^2\cdot\sin(\theta)^2}} = \frac{-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)}{\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}$$

so... $$\frac{d}{dt}h(\theta) = \text{Part\_1} + \text{Part\_2} = (-r\cdot\omega\cdot\sin(\theta)) + \frac{-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)}{\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}$$

Example: **Assume a 10,000 RPM:** $\omega := \frac{10000 \cdot 360 \cdot deg}{min}$ $\omega = 1047 \frac{1}{s}$ and break the velocity equation into parts

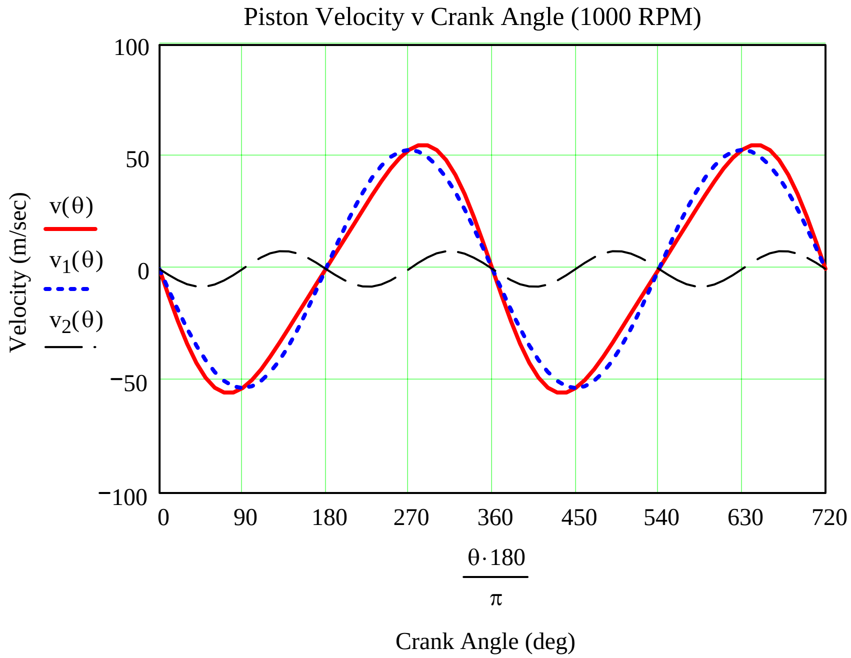
$$v_1(\theta) := -r \cdot \omega \cdot \sin(\theta)$$

$$v_2(\theta) := \frac{-r^2 \cdot \omega \cdot \sin(\theta) \cdot \cos(\theta)}{\sqrt{L^2 - r^2 \cdot \sin(\theta)^2}}$$

$$v(\theta) := (-r \cdot \omega \cdot \sin(\theta)) + \frac{-r^2 \cdot \omega \cdot \sin(\theta) \cdot \cos(\theta)}{\sqrt{L^2 - r^2 \cdot \sin(\theta)^2}}$$

(the complete velocity equation)

Graphing this function over the domain: $\theta := 0 \cdot deg, 10 \cdot deg .. 720 \cdot deg$



Maximum velocity occurs at 74.84 degrees and 285.16 degrees after Top Dead Center as shown in the calculations (roots of the velocity curve) below:

$$v_{max}(\theta) := \frac{d}{d\theta} v(\theta)$$

using an initial guess of: $\theta := 85 \cdot deg$ $\quad root(v_{max}(\theta), \theta) = 74.84\,deg$

using an initial guess of: $\theta := 300 \cdot deg$ $\quad root(v_{max}(\theta), \theta) = 285.16\,deg$

Derive the piston acceleration (α) equation. Acceleration is just the time rate of change of the velocity v(θ).

Given: $$v(\theta) = (-r\cdot\omega\cdot\sin(\theta)) + \frac{-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)}{\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}$$

Taking the derivative of the velocity with respect to time:

$$\frac{d}{dt}v(\theta) = \frac{d}{dt}(-r\cdot\omega\cdot\sin(\theta)) + \frac{d}{dt}\frac{-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)}{\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}$$

remember: $\alpha = \frac{d^2}{dt^2}\theta = \frac{d}{dt}\omega$ (the time rate of change of ω (RPM) is angular acceleration - "reving the engine")

assume: $\alpha := 0\cdot\frac{rad}{sec^2}$ (a constant RPM - no "reving" the engine)

Breaking the derivation into parts:

Part 1 $$\frac{d}{dt}(-r\cdot\omega\cdot\sin(\theta)) = \left[-r\cdot\omega\cdot\left(\frac{d}{dt}\sin(\theta)\right)\right] + -r\cdot\sin(\theta)\cdot\frac{d}{dt}\omega + \omega\cdot\sin(\theta)\cdot\frac{d}{dt}-r = (-r\cdot\omega\cdot\omega\cdot\cos(\theta)) + -r\cdot\sin(\theta)\cdot\alpha + 0 = -r\cdot\omega^2\cdot\cos(\theta) - r\cdot\sin(\theta)\cdot\alpha$$

Letting: $$p_1(\theta) := -r\cdot\omega^2\cdot\cos(\theta) - r\cdot\sin(\theta)\cdot\alpha$$

Part 2

$$\frac{d}{dt}\frac{-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)}{\sqrt{L^2 - r^2\cdot\sin(\theta)^2}} = \frac{d}{dt}\left[\left(-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\right)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\right]$$

Part 2.1

$$-r^2\cdot\omega\cdot\sin(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\frac{d}{dt}\cos(\theta) = -r^2\cdot\omega\cdot\sin(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot(-\omega\cdot\sin(\theta))$$

Letting: $p_{21}(\theta) := -r^2\cdot\omega\cdot\sin(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot(-\omega\cdot\sin(\theta))$

Part 2.2

$$-r^2\cdot\omega\cdot\cos(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\left(\frac{d}{dt}\sin(\theta)\right) = -r^2\cdot\omega\cdot\cos(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot(\omega\cdot\cos(\theta))$$

Letting: $p_{22}(\theta) := -r^2\cdot\omega\cdot\cos(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot(\omega\cdot\cos(\theta))$

Part 2.3

$$-r^2\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{d}{dt}\omega = -r^2\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\sin(\theta)\cdot\cos(\theta)\cdot\alpha$$

Letting: $p_{23}(\theta) := -r^2\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\sin(\theta)\cdot\cos(\theta)\cdot\alpha$

Part 2.4 $(\omega\cdot\sin(\theta)\cdot\cos(\theta))\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\frac{d}{dt}r = 0$ **(since crank throw does not change with time)**

Letting: $p_{24}(\theta) := 0\cdot\frac{ft}{sec^2}$

Part 2.5

$$-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{d}{dt}\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}} = -r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{-1}{2}\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-3}{2}}\cdot\frac{d}{dt}\left(L^2 - r^2\cdot\sin(\theta)^2\right)$$

$$-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{d}{dt}\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}} = -r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{-1}{2}\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-3}{2}}\cdot\left[0 + (-2)\cdot\left(r^2\right)\cdot\left[\sin(\theta)\cdot\left[(\omega)\cdot\cos(\theta)\right]\right]\right]$$

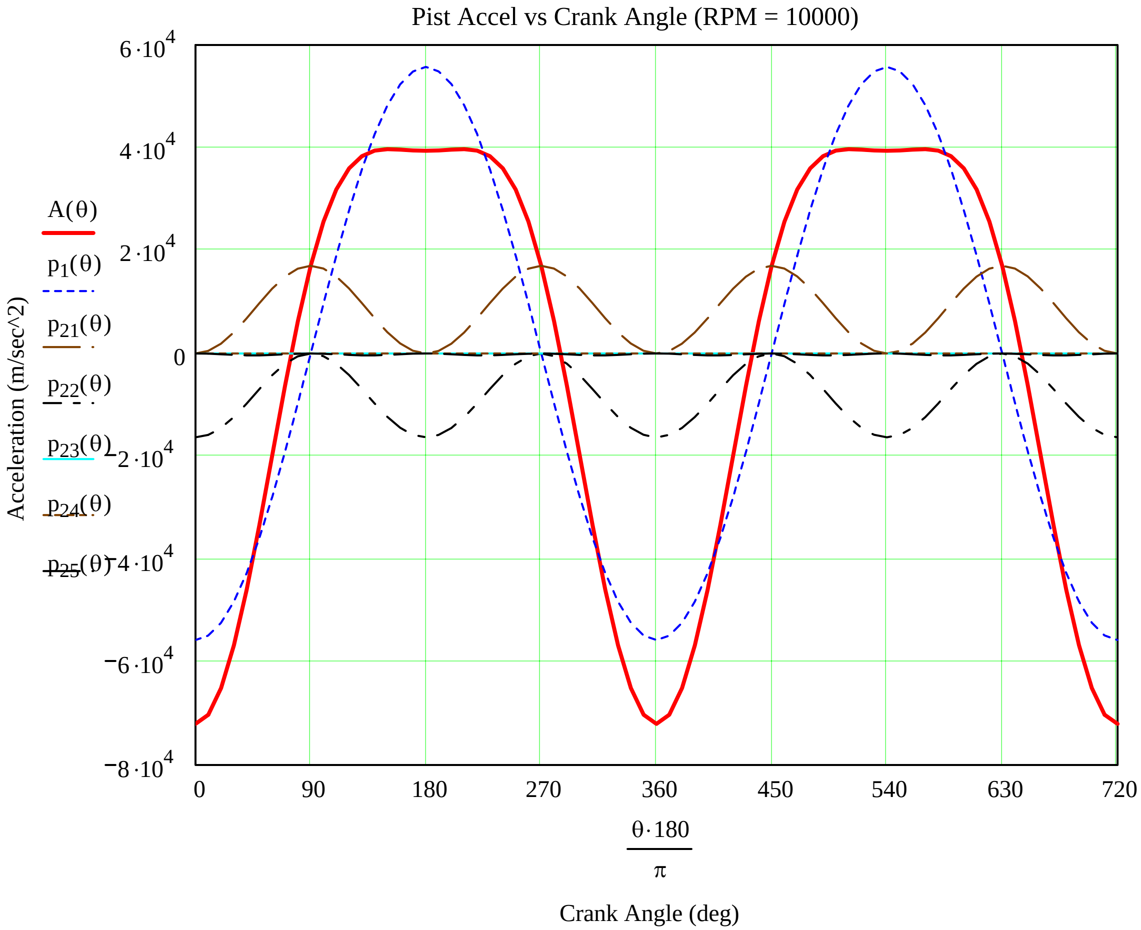
Letting: $p_{25}(\theta) := -r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{-1}{2}\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-3}{2}}\cdot\left[0 + (-2)\cdot\left(r^2\right)\cdot\left[\sin(\theta)\cdot\left[(\omega)\cdot\cos(\theta)\right]\right]\right]$

Simplifying: $p_{25}(\theta) := r^4\cdot\omega^2\cdot\left(-1 + \cos(\theta)^2\right)\cdot\frac{\cos(\theta)^2}{\left(L^2 - r^2 + r^2\cdot\cos(\theta)^2\right)^{\frac{3}{2}}}$

Adding the parts together to get the complete acceleration function and graphing it it over the domain:

$\theta := 0 \cdot deg, 10 \cdot deg .. 720 \cdot deg$

$$A(\theta) := p_1(\theta) + p_{21}(\theta) + p_{22}(\theta) + p_{23}(\theta) + p_{24}(\theta) + p_{25}(\theta)$$



example acceleration at 220 degrees past TDC:

$$A(220 \cdot deg) = 39436 \, \frac{m}{s^2}$$

example g calculation:

$$\frac{A(220 \cdot deg)}{g} = 4021$$

Connecting rod stress analysis:

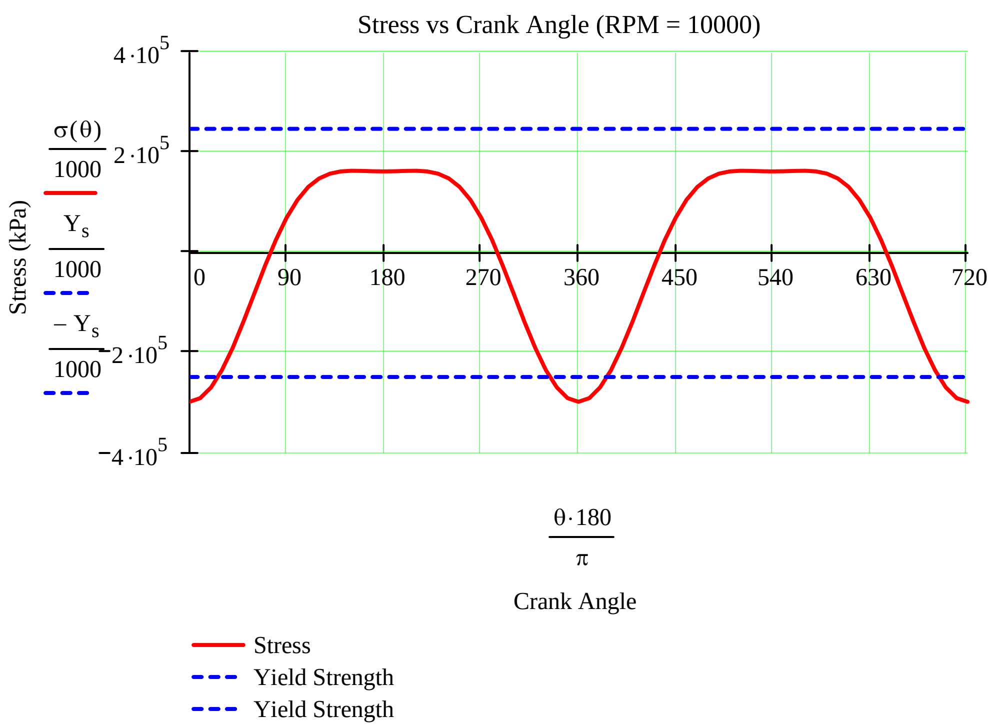
Define: $kPa := 1000 \cdot Pa$

graphing domain: $\theta := 0 \cdot deg, 10 \cdot deg .. 720 \cdot deg$

Yield Strength of rod steel: $Y_s := 36000 \cdot psi$ $\quad Y_s = 248211\, kPa$

Variables: $cross_{sec} := 0.51 \cdot in^2$ $\quad Mass_{piston} := 3 \cdot lb$ $\quad \omega = 1047 \frac{1}{s}$ **10,000 RPM**

Using Newton's 2nd Law: $F = ma$ $\quad F(\theta) := Mass_{piston} \cdot A(\theta)$ $\quad \sigma(\theta) := \frac{F(\theta)}{cross_{sec}}$



Finding failure crank angles using Mathcad's root finding function with initial guesses of:

$\theta := -10 \cdot deg$ $\quad root(\sigma(\theta) + Y_s, \theta) = -26\, deg$

$\theta := 10 \cdot deg$ $\quad root(\sigma(\theta) + Y_s, \theta) = 26\, deg$

Conclusion: For the given geometry and parameters (RPM = 10,000), the failure will occur near TDC (from 26 degrees before until 26 degrees after TDC)

Graphing in 3D and contour plot the piston acceleration (α) equation as a function of crank angle (θ) <u>and</u> engine RPM (ω). This will be a 3-D analysis!

Remembering: $$v(\theta,\omega) = (-r\cdot\omega\cdot\sin(\theta)) + \frac{-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)}{\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}$$

Taking the derivative of the velocity with respect to time: $$\frac{d}{dt}v(\theta,\omega) = \frac{d}{dt}(-r\cdot\omega\cdot\sin(\theta)) + \frac{d}{dt}\frac{-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)}{\sqrt{L^2 - r^2\cdot\sin(\theta)^2}}$$

remember: $$\alpha = \frac{d^2}{dt^2}\theta = \frac{d}{dt}\omega$$

assume: $\alpha := 0\cdot\frac{rad}{sec^2}$ (a constant RPM)

Breaking the derivation into parts:

Part 1 $$\frac{d}{dt}(-r\cdot\omega\cdot\sin(\theta)) = \left[-r\cdot\omega\cdot\left(\frac{d}{dt}\sin(\theta)\right)\right] + -r\cdot\sin(\theta)\cdot\frac{d}{dt}\omega + \omega\cdot\sin(\theta)\cdot\frac{d}{dt}-r = (-r\cdot\omega\cdot\omega\cdot\cos(\theta)) + -r\cdot\sin(\theta)\cdot\alpha + 0 = -r\cdot\omega^2\cdot\cos(\theta) - r\cdot\sin(\theta)\cdot\alpha$$

Letting: $p_1(\theta,\omega) := -r\cdot\omega^2\cdot\cos(\theta) - r\cdot\sin(\theta)\cdot\alpha$

Part 2

$$\frac{d}{dt}\frac{-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)}{\sqrt{L^2 - r^2\cdot\sin(\theta)^2}} = \frac{d}{dt}\left[\left(-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\right)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\right]$$

Part 2.1

$$-r^2\cdot\omega\cdot\sin(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\frac{d}{dt}\cos(\theta) = -r^2\cdot\omega\cdot\sin(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot(-\omega\cdot\sin(\theta))$$

Letting: $p_{21}(\theta,\omega) := -r^2\cdot\omega\cdot\sin(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot(-\omega\cdot\sin(\theta))$

Part 2.2

$$-r^2\cdot\omega\cdot\cos(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\left(\frac{d}{dt}\sin(\theta)\right) = -r^2\cdot\omega\cdot\cos(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot(\omega\cdot\cos(\theta))$$

Letting: $p_{22}(\theta,\omega) := -r^2\cdot\omega\cdot\cos(\theta)\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot(\omega\cdot\cos(\theta))$

Part 2.3

$$\left[-r^2\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{d}{dt}\omega\right] = -r^2\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\sin(\theta)\cdot\cos(\theta)\cdot\alpha$$

Letting: $p_{23}(\theta,\omega) := -r^2\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\sin(\theta)\cdot\cos(\theta)\cdot\alpha$

Part 2.4 $(\omega\cdot\sin(\theta)\cdot\cos(\theta))\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\cdot\frac{d}{dt}-r = 0$ **(since crank throw does not change with time)**

so... letting: $p_{24}(\theta,\omega) := 0\cdot\frac{ft}{sec^2}$

Part 2.5

$$\left[-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{d}{dt}\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\right] = -r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{-1}{2}\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-3}{2}}\cdot\frac{d}{dt}\left(L^2 - r^2\cdot\sin(\theta)^2\right)$$

$$\left[-r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{d}{dt}\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-1}{2}}\right] = -r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{-1}{2}\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-3}{2}}\cdot\left[0 + (-2)\cdot\left(r^2\right)\cdot\left[\sin(\theta)\cdot\left[(\omega)\cdot\cos(\theta)\right]\right]\right]$$

Letting: $p_{25}(\theta) := -r^2\cdot\omega\cdot\sin(\theta)\cdot\cos(\theta)\cdot\frac{-1}{2}\cdot\left(L^2 - r^2\cdot\sin(\theta)^2\right)^{\frac{-3}{2}}\cdot\left[0 + (-2)\cdot\left(r^2\right)\cdot\left[\sin(\theta)\cdot\left[(\omega)\cdot\cos(\theta)\right]\right]\right]$

Simplifying: $$p_{25}(\theta,\omega) := -\left[-r^4\cdot\omega^2\cdot\left(-1+\cos(\theta)^2\right)\cdot\frac{\cos(\theta)^2}{\left(L^2-r^2+r^2\cdot\cos(\theta)^2\right)^{\frac{3}{2}}}\right]$$

Adding the parts together to get the single (and complete) acceleration function as a function of crank angle (θ) and engine RPM (ω) [and graphing it it over the domain shown to the right] yields:

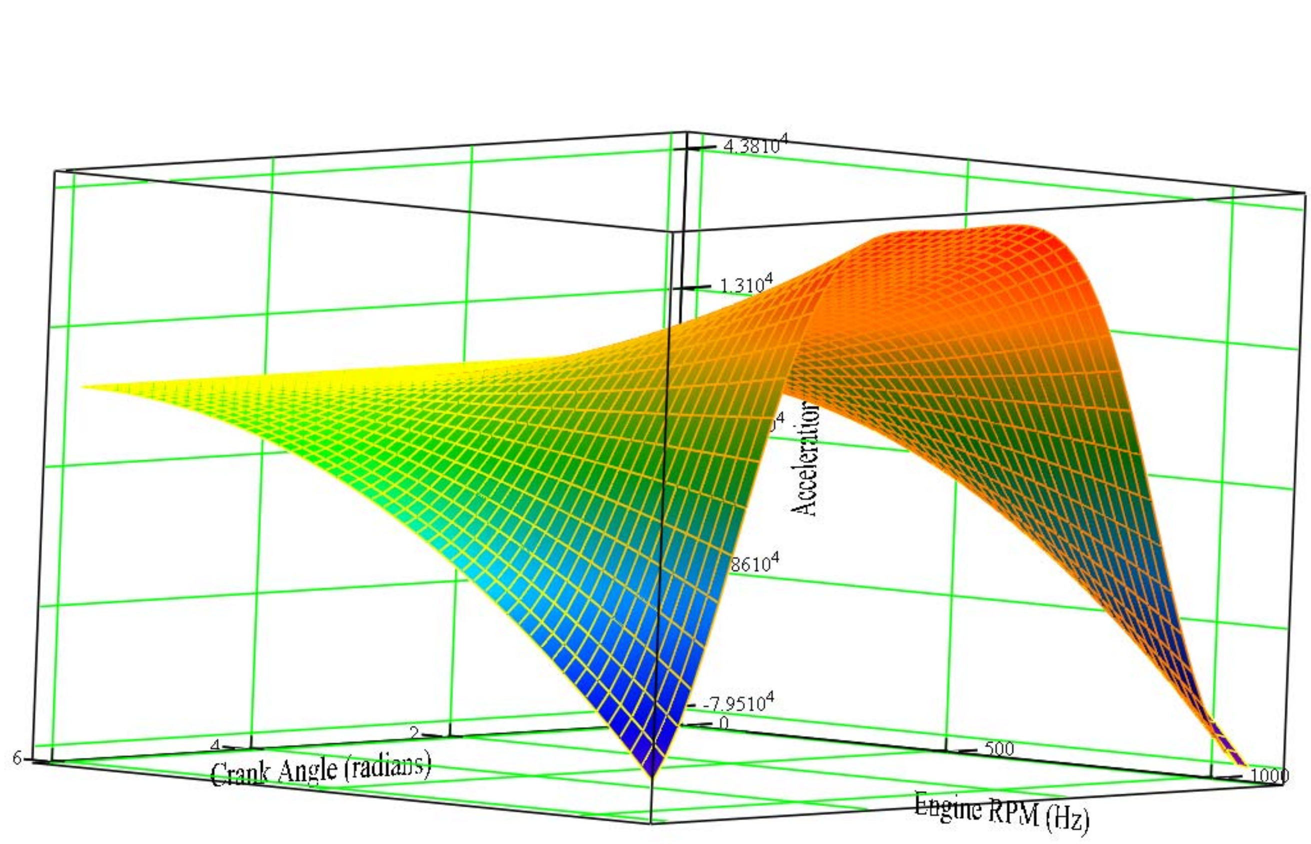
$0 \le \theta \le 6$ (radians)

$0 \le \omega \le 1100$ (Hz) where: 1100 Hz = 10504 RPM

$$A(\theta,\omega) := p_1(\theta,\omega) + p_{21}(\theta,\omega) + p_{22}(\theta,\omega) + p_{23}(\theta,\omega) + p_{24}(\theta,\omega) + p_{25}(\theta,\omega)$$

3-D Plot

Piston Acceleration vs. Crank Angle and Engine Speed



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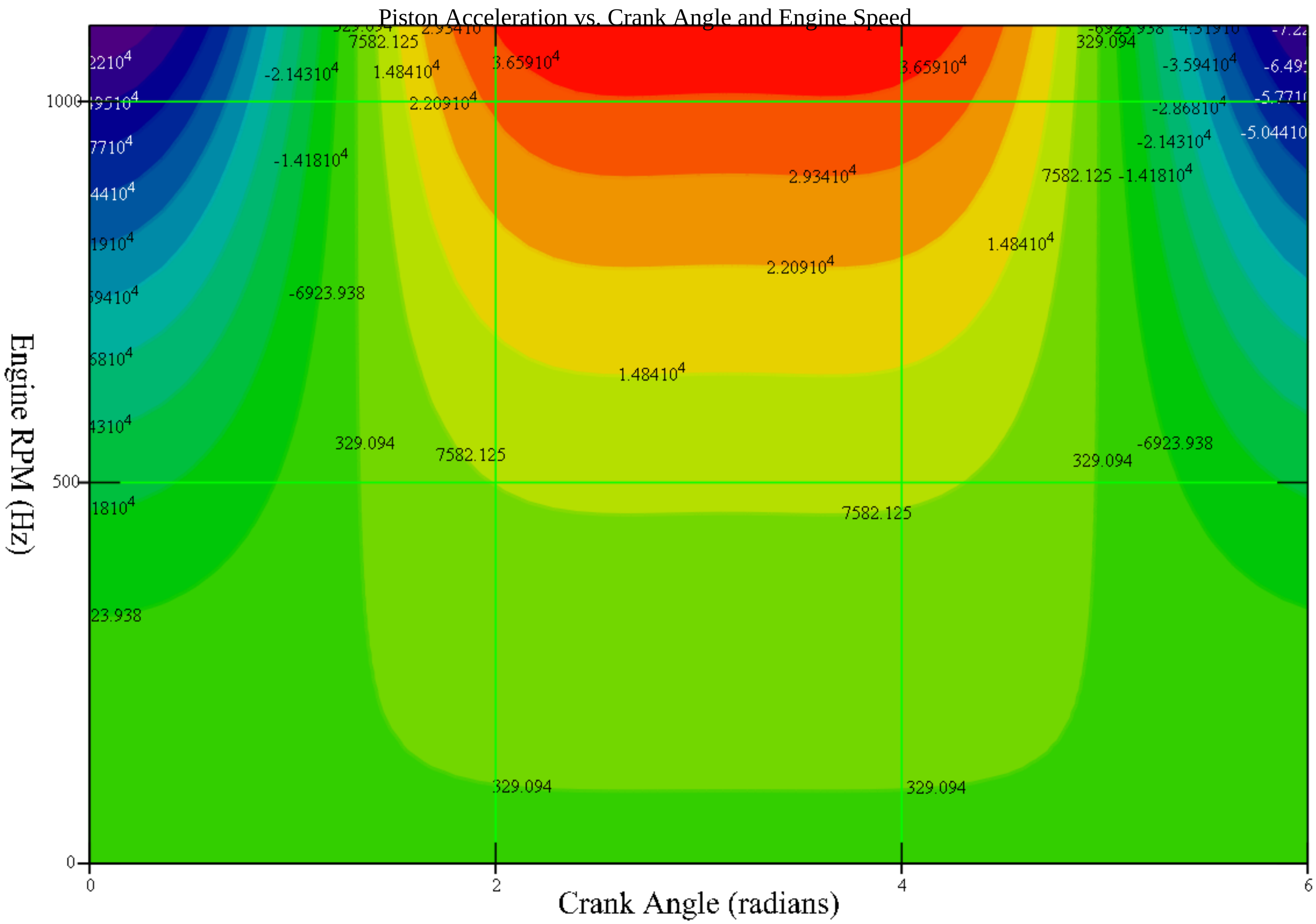
Contour Plot of piston acceleration as a function of crank angle and engine speed (RPM)



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