M. Heinen
2/15/2006

This analysis investigates the **stochastic IRR problem** (shown to the right) where A_0, A_1,..., A_n are cash flows (random variables) derived in this analysis from known exponential distributions. The goal is to develop the probability density function (pdf) and cummulative distribution function (cdf) for the internal rate of return (i) for this investment. A **deterministic** example of IRR is shown below.

$$0 = A_0 + \frac{A_1}{(1+i)^1} + \frac{A_2}{(1+i)^2} + ... + \frac{A_n}{(1+i)^n}$$

Given this cash flow $P := \begin{pmatrix} -1000 \\ 1300 \\ -400 \\ 200 \end{pmatrix}$

IRR determined by finding the roots of: $f(i) := P_0 + P_1 \cdot \frac{1}{1+i} + P_2 \cdot \left(\frac{1}{1+i}\right)^2 + P_3 \cdot \left(\frac{1}{1+i}\right)^3$

Initial guess: $i := .5$ $\quad i_r := \text{root}(f(i), i)$

IRR: $i_r = 0.102$

Project balances given by:

$$PB(N) := \sum_{n=0}^{N} P_n \cdot (1+i)^{N-n}$$

N =	PB(N) =
0	-1000
1	198
2	-182
3	-0

Only the simplest form of this problem will be considered where: $0 = A_0 + \frac{A_1}{1+i}$ (where i is the interest rate)

The PDF, CDF and mean for two cases of this basic problem are determined.

Example

Page 3 **Case 1** - A_0 is a constant (fixed initial investment) and A_1 is a random variable derived from an exponential distribution with a know mean.

$$0 = -1000 + \frac{A_1}{1+i}$$

Page 9 **Case 2** - A_0 and A_1 are both random variables. Two examples are considered:

$$0 = A_0 + \frac{A_1}{1+i}$$

Page 14 **Example 1** - both random variables distributions are derived from exponential distributions having the same means.

Page 21 **Example 2** - random variables are derived from exponential distributions with differing means.

See the page 27 for a summary of the equations for resulting distribution PDFs, CDFs, and means as a function of the interest rate (i).

Case 1 - A_0 is a constant

For the problem: $$0 = A_0 + \frac{A_1}{1+i}$$

Letting y = 1 + i, A_0 = k, A_1 = x, where k is a constant, x and y are RVs, the problem becomes: $$0 = k + \frac{x}{y} \quad \text{or} \quad y = \frac{-x}{k}$$

Can y = 1 + i ?
Yes if the relationship is multiplied by or is added to by a constant... See Shooman's text, page 394, # 1 and # 2.
To get f(i), see transformations at end of this Case 1 analysis.

Determine the probability density function (PDF) for the random variable y given that x is derived from an exponential distribution. $$y = \frac{-x}{k}$$

Using the fundamental theorem (From A. Papoulis text, page 95, Probability, Random Variables, and Stochastic Processes, 1984, ISBN 0-07-048468-6):

Given a probability density function $f_x(\mathbf{x})$, to find $f_y(\mathbf{y})$ for a specific y, we solve the equation y = g(x), where g(x) denotes the relationship between the random variables. Denoting it's real roots by x_n $$y = g(x_1) = ... = g(x_n)$$

Letting $$g'(x) = \frac{d}{dx}g(x) \qquad f_y(y) = \frac{f_x(x_1)}{|g'(x_1)|} + \frac{f_x(x_2)}{|g'(x_2)|} + ... + \frac{f_x(x_n)}{|g'(x_n)|}$$

Determine **y,** given **x** and **y** are related random variables (with **x** derived from some distribution and k is a constant).

For **Case 1** $y = \frac{-x}{k}$ **(remembering for our IRR problem y = 1 + i)**

thus $$g(x) = \frac{-x}{k} \qquad \frac{d}{dx}g(x) = \frac{-1}{k} \quad \text{or} \quad g'(x) = \frac{-1}{k} \qquad |g'(x)| = \left|\frac{1}{k}\right|$$

or $x = -k \cdot y$

The equation y = - x / k has a **single** solution at x = - k * y. Applying the fundamental theorem:

$$f_y(y) = \frac{f_x(x)}{|g'(x)|} = \frac{f_x(x)}{\left|\frac{1}{k}\right|} = k \cdot f_x(x)$$

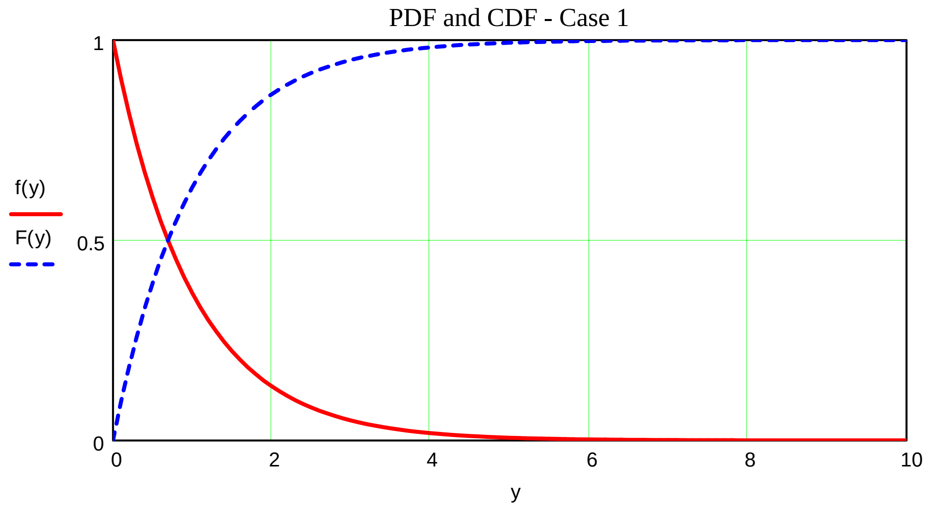
For the exponential distribution: $f_x(x) = \lambda \cdot e^{-\lambda \cdot x}$

The PDF for y becomes: $f(y) = |k| \cdot \left[\lambda \cdot e^{-\lambda \cdot (-k \cdot y)}\right] = |k| \cdot \lambda \cdot e^{(k \cdot \lambda \cdot y)}$

Checking to seem if f(y) is a valid PDF for 0 < y < ∞ where k is some negative constant (since for our IRR Problem, k represents A_o or some initial investment (negative) at the first period 0):

$y := 0, .1 .. 10$ $\lambda := 1$ $k := -1$ $f(y) := |k| \cdot \left[\lambda \cdot e^{-\lambda \cdot (-k \cdot y)}\right]$ $F(y) := \int_0^y f(y)\, dy$

Integrating the PDF numerically and checking the result using a large y.



$$\int_0^{\infty} f(y)\, dy = 1$$

Computing the mean for this specific problem **(remembering λ = 1 and k = -1)**:

$$\mu := \int_0^{\infty} y \cdot f(y)\, dy$$

$\mu = 1$ (when y = 1 + i)

Changing the distribution from f(y) to f(i) where y = 1 + i where i is the interest rate (per the discussion 3/25/02).

Case 1 - A_o is a constant and A_1 is exponentially distributed.

$\lambda := 1 \qquad k := -1 \qquad f(z) = |k| \cdot \lambda \cdot e^{(k \cdot \lambda \cdot y)} \qquad F(z) = 1 - \int_0^z f(z)\, dz$

original limits of integration $\quad 0 < z < \infty$

substitution since z =1 + i $\quad 0 < i + 1 < \infty$

new limits for i $\quad -1 < i < \infty$

remembering $\quad F(z) = \int_0^z |k| \cdot \lambda \cdot e^{(k \cdot \lambda \cdot y)}\, dz$

making a change in variables: $\quad F(i) = \int_{-1}^{i} |k| \cdot \left[\lambda \cdot e^{\lambda \cdot k \cdot (1+i)}\right] di$

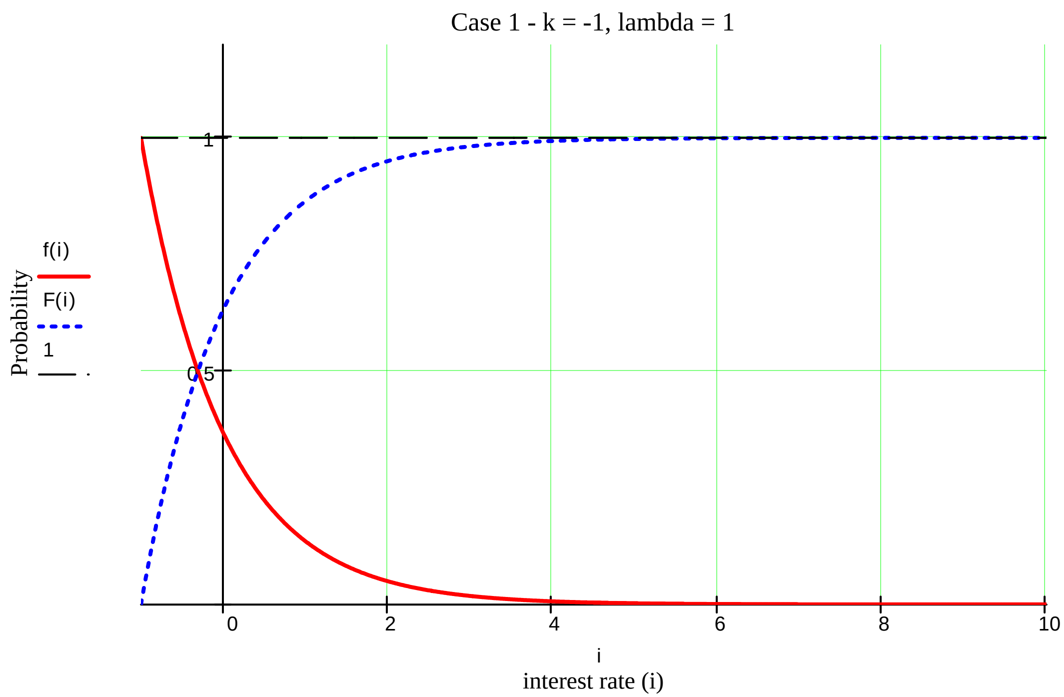
where the indefinite integral is: $\quad \int |k| \cdot \left[\lambda \cdot e^{\lambda \cdot k \cdot (1+i)}\right] di = \frac{|k|}{k} \cdot e^{[\lambda \cdot k \cdot (1+i)]}$

calculating the CDF for i using the new limits $\quad F(i) = \int_{-1}^{i} |k| \cdot \left[\lambda \cdot e^{\lambda \cdot k \cdot (1+i)}\right] di = \frac{|k|}{k} \cdot e^{[\lambda \cdot k \cdot (1+i)]} - \frac{|k|}{k} \cdot e^{[\lambda \cdot k \cdot [1+(-1)]]}$

the CDF simplifies to: $F(i) := |k| \cdot \frac{[\exp[\lambda \cdot k \cdot (1+i)] - 1]}{k}$ (CDF)

calculating the PDF for i $f(i) = \frac{d}{di}(F(i)) = \frac{d}{di}\left[1 - |k| \cdot \frac{[\exp[\lambda \cdot k \cdot (1+i)] - 1]}{k}\right]$ or $f(i) := |k| \cdot \lambda \cdot \exp[\lambda \cdot k \cdot (1+i)]$ (PDF)

remembering λ = 1 and k = -1 and graphing the PDF and CDF over the domain: $i := -1, -.99 .. 10$



Confirming the CDF = 1:

$$\int_{-1}^{\infty} f(i)\, di = 1$$

$$\mu := \int_{-1}^{\infty} i \cdot f(i)\, di$$

$$\mu = -8.316 \times 10^{-13}$$

(μ = 0 since k = -1 and λ = 1)

Calculating the mean for Case 1.

Using the pdf developed on the previous page: $f(i) := |k| \cdot \lambda \cdot \exp[\lambda \cdot k \cdot (1 + i)]$

The mean is defined as: $\mu = \int_{-1}^{\infty} i \cdot [\,|k| \cdot \lambda \cdot \exp[\lambda \cdot k \cdot (1 + i)]\,]\, di$

Integrating yields:

$$\int_{-1}^{\infty} i \cdot [\,|k| \cdot \lambda \cdot \exp[\lambda \cdot k \cdot (1 + i)]\,]\, di = \frac{[\exp(\lambda \cdot k \cdot \infty + \lambda \cdot k) \cdot (\lambda \cdot k \cdot \infty + \lambda \cdot k) - \exp(\lambda \cdot k \cdot \infty + \lambda \cdot k) - \exp(\lambda \cdot k \cdot \infty + \lambda \cdot k) \cdot \lambda \cdot k]}{\lambda \cdot k^2} \cdot |k| - \left[\frac{[\exp[\lambda \cdot k \cdot (-1) + \lambda \cdot k] \cdot [\lambda \cdot k \cdot (-1) + \lambda \cdot k] - \exp[\lambda \cdot k \cdot (-1) + \lambda \cdot k] - \exp[\lambda \cdot k \cdot (-1) + \lambda \cdot k] \cdot \lambda \cdot k]}{\lambda \cdot k^2} \cdot |k| \right]$$

or $\mu = |k| \cdot \frac{(\exp(\lambda \cdot k \cdot \infty) \cdot \lambda \cdot k \cdot \infty - \exp(\lambda \cdot k \cdot \infty) + 1 + \lambda \cdot k)}{\lambda \cdot k^2}$

or $\mu = 0$

Case 1 - What does this all mean? Remembering Case 1 is the problem to the right where A_o is a constant (a known investment) and A_1 is a single payment (to repay the investment) derived from an exponential distribution with a known mean. Consider the following example:

$$0 = A_0 + \frac{A_1}{1+i}$$

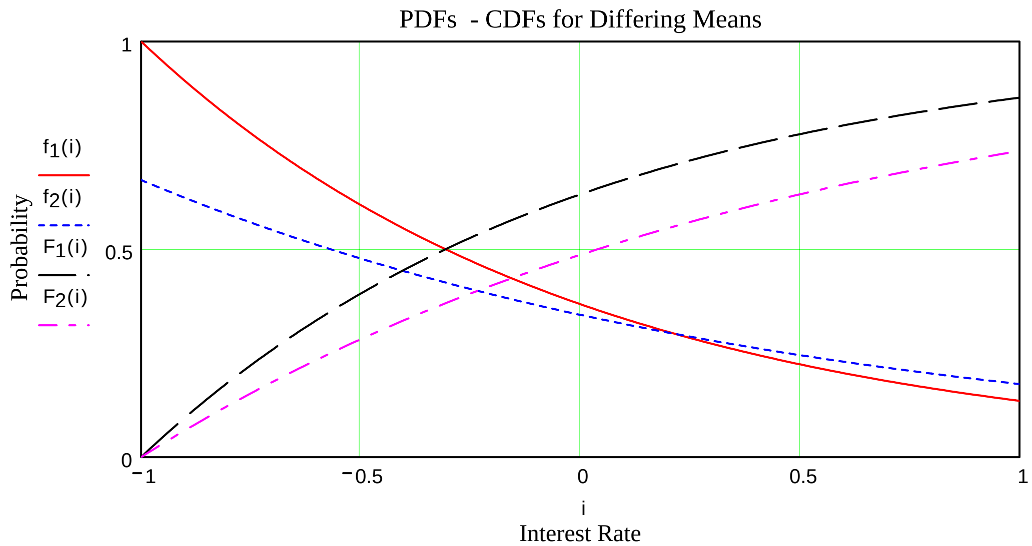
Suppose two companies desire a \$10,000 investment, each with different mean for a single payment derived from exponential distribution:

Money invested $k := -10000$

Company A $\mu_1 := 10000$ $\lambda_1 := \frac{1}{\mu_1}$ $f_1(i) := |k| \cdot \lambda_1 \cdot \exp[\lambda_1 \cdot k \cdot (1+i)]$ $F_1(i) := \int_{-1}^{i} f_1(i)\, di$

Company B $\mu_2 := 15000$ $\lambda_2 := \frac{1}{\mu_2}$ $f_2(i) := |k| \cdot \lambda_2 \cdot \exp[\lambda_2 \cdot k \cdot (1+i)]$ $F_2(i) := \int_{-1}^{i} f_2(i)\, di$

$i := -1, -.99 .. 1$



Assume you wish to have an IRR of **exactly** 20%.

$i := 0.20$	**Probability**
Company A	$f_1(i) = 0.301$
Company B	$f_2(i) = 0.3$

Assume the IRR must be **greater than** 20%.

$i := 0.20$	**Probability**
Company A	$1 - F_1(i) = 0.301$
Company B	$1 - F_2(i) = 0.449$

Company B appears to be the better choice to invest in! Of course, we knew that since it had a greater mean.

Case 2 - A_0 and A1 are both random variables drawn from different exponential distributions.

For the problem: $0 = A_0 + \frac{A_1}{1+i}$

Letting x = (1+i), where A_0 and A_1 are RVs, the problem becomes: $0 = A_0 + \frac{A_1}{x}$

Determine the probability density function (PDF) for the random variable x given that A_0 and A_1 are derived from exponential distributions. $x = \frac{-A_1}{A_0}$

Since this problem requires the development of a PDF that is **the quotient of two random variables**, a discussion of the theory is begins on the next page.

This theory is extrapolated from:

B. V. Gnedenko, The Theory of Probability; Chelsea Publishing Company, 1962, pages 185-186

A. Papoulis, Probability, Random Variables, and Stochastic Processes; McGraw-Hill Book Company, 1965

Analysis of the quotient of two random variables

Determine the density function f(z) representing the quotient of the two independent random variables x and y derived from continuous density functions $f_1(x)$ and $f_2(y)$.

The region D_z, of the xy plane such that: $\frac{x}{y} \le z$

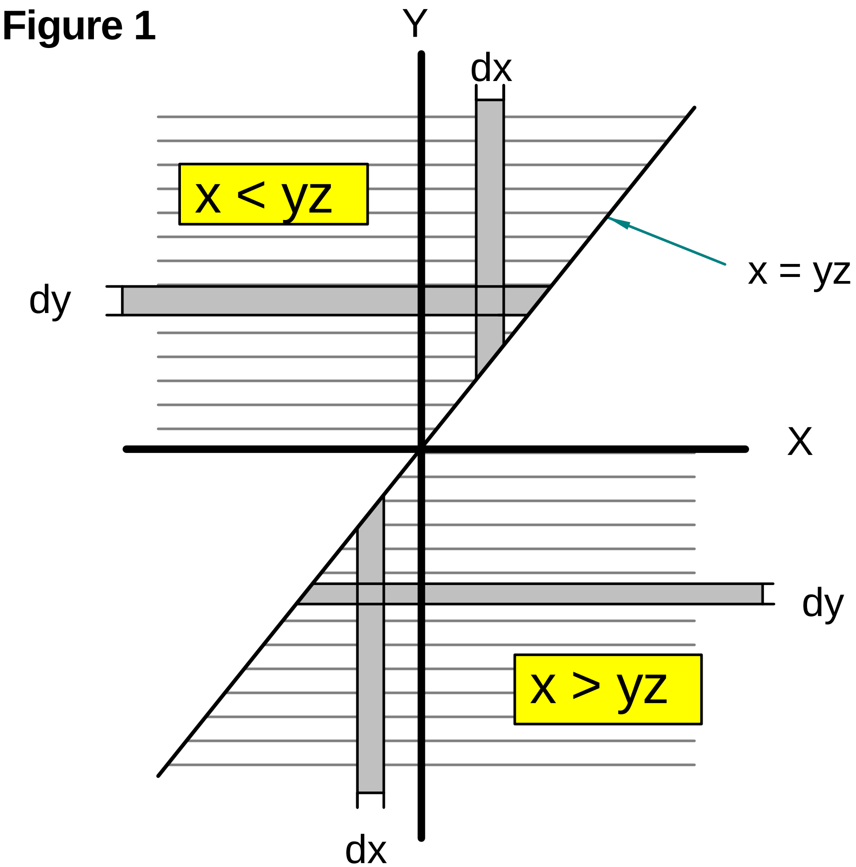
is the lined area of Figure 1 shown to the right because:

If $y > 0$ then $x \le yz$

If $y < 0$ then $x \ge yz$

For the two random variables x and y, the masses in D_z are just the just volumes under f(z) as seen in **Figure 2**. The volumes under f(z) are just the CDF - cumulative distribution function F(z).

Figure 1



from Figures 1 and 2

$$F_z(z) = \int_0^{\infty}\int_{-\infty}^{x} f(x,y)\,dx\,dy + \int_{-\infty}^{0}\int_{x}^{\infty} f(x,y)\,dx\,dy \qquad [1]$$

since $x = yz$

$$F_z(z) = \int_{0}^{\infty}\int_{-\infty}^{zy} f(x,y)\,dx\,dy + \int_{-\infty}^{0}\int_{zy}^{\infty} f(x,y)\,dx\,dy \qquad [2]$$

since $f_1(x)$ and $f_2(y)$ are **independent**

$$F_z(z) = \int_{0}^{\infty}\int_{-\infty}^{zy} f_1(x)\cdot f_2(y)\,dx\,dy + \int_{-\infty}^{0}\int_{zy}^{\infty} f_1(x)\cdot f_2(y)\,dx\,dy \qquad [3]$$

integrating the inner most terms with respect to x

$$F_z(z) = \int_{0}^{\infty} \left(F_1(zy) - F_1(-\infty)\right)\cdot f_2(y)\,dy + \int_{-\infty}^{0} \left(F_1(\infty) - F_1(zy)\right)\cdot f_2(y)\,dy \qquad [4]$$

but since the probabilities **must sum to 1**

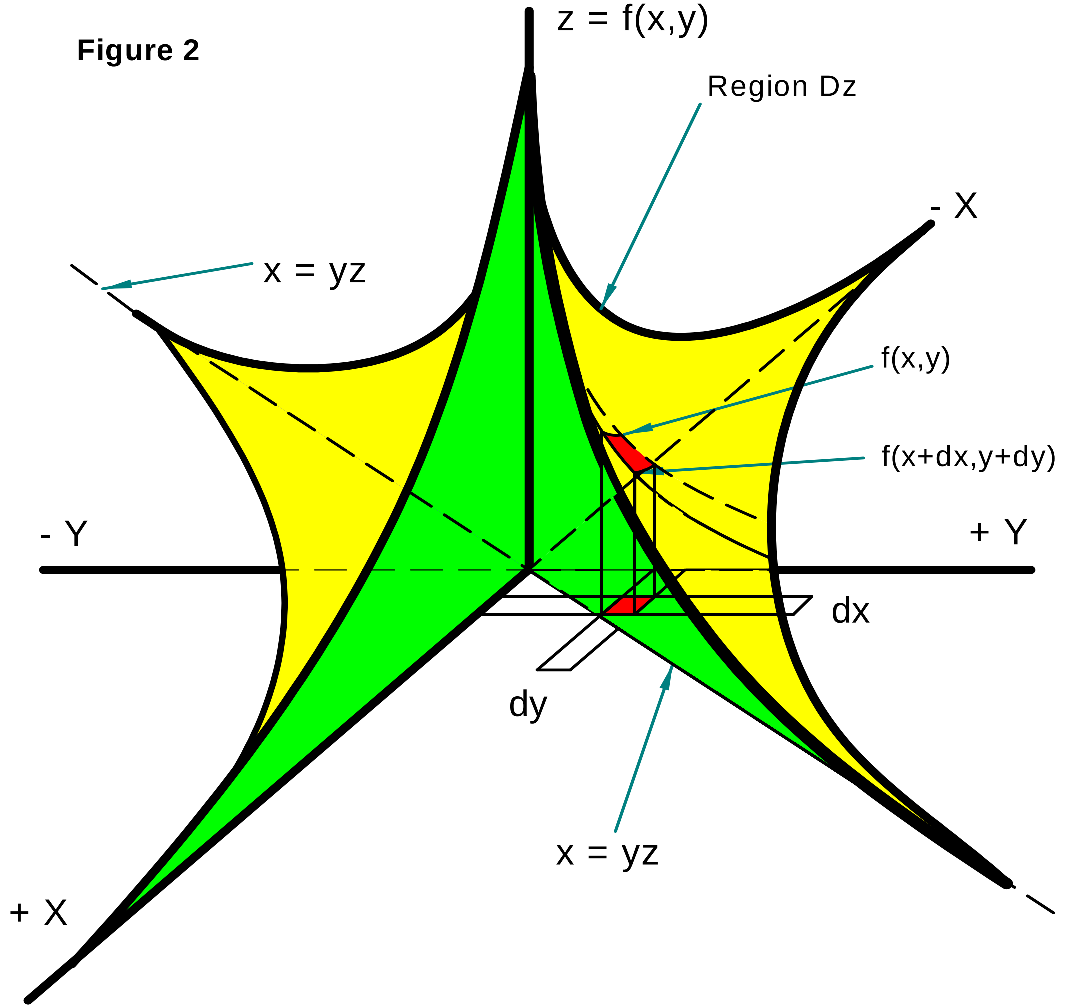
$$F_z(z) = \int_{0}^{\infty} \left(F_1(zy) - F_1(-\infty)\right)\cdot f_2(y)\,dy + \int_{-\infty}^{0} \left[1 - \left(F_1(zy) - F_1(-\infty)\right)\right]\cdot f_2(y)\,dy \qquad [5]$$

differentiating with respect to z to get the PDF probability density function f(z):

$$f_z(z) = \int_{0}^{\infty} z\,f_1(zy)\cdot f_2(y)\,dy - \int_{-\infty}^{0} z\cdot f_1(zy)\cdot f_2(y)\,dy \qquad [6]$$

Equation [6] provides the PDF (probability density function) for the quotient of 2 random variables x and y derived from continuous, independent density functions $f_1(x)$ and $f_2(y)$.

Figure 2



Determine the probability density function $f_z(z)$ representing the ***quotient*** of two random variables ε and η derived respectively from *independent* exponential density functions.

$$f_1(x) = \lambda_1 \cdot e^{-\lambda_1 \cdot x} \qquad f_2(y) = \lambda_2 \cdot e^{-\lambda_2 \cdot y}$$

Remembering, for the IRR problem: z = 1 + i (where i is the interest rate).

Equation [6] derived the general density function for the quotient of two random variables:

$$f_z(z) = \int_0^{\infty} y\, f_1(zy)\cdot f_2(y)\, dy - \int_{-\infty}^{0} y\cdot f_1(zy)\cdot f_2(y)\, dy$$

substitution yields:

$$f_z(z) = \int_0^{\infty} y\left(\lambda_1\cdot e^{-\lambda_1\cdot zy}\right)\cdot\left(\lambda_2\cdot e^{-\lambda_2\cdot y}\right) dy - \int_{-\infty}^{0} y\cdot\left(\lambda_1\cdot e^{-\lambda_1\cdot zy}\right)\cdot\left(\lambda_2\cdot e^{-\lambda_2\cdot y}\right) dy \qquad [7]$$

the indefinite integral of $y\cdot\left(\lambda_1\cdot e^{-\lambda_1\cdot zy}\right)\cdot\left(\lambda_2\cdot e^{-\lambda_2\cdot y}\right)$

is

$$\int y\cdot\lambda_1\cdot e^{-\lambda_1\cdot z\cdot y}\cdot\lambda_2\cdot e^{-\lambda_2\cdot y}\, dy = \frac{\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot y\cdot \exp\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot y\right] - \exp\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot y\right]\right]}{\left(-\lambda_1\cdot z-\lambda_2\right)^2}\cdot\lambda_1\cdot\lambda_2 \qquad [7a]$$

Example 1: Letting $\lambda_1 = \lambda_2 = 1$ the indefinite integral of equation [7a] becomes:

$$-\exp[-(z+1)\cdot y]\cdot\frac{(z\cdot y+y+1)}{(z+1)^2}$$

the definite integral for $f_z(z)$ is:

$$f_z(z) = \left[-\exp[-(z+1)\cdot\infty]\cdot\frac{(\infty\cdot z+\infty+1)}{(z+1)^2} - -\exp[-(z+1)\cdot 0]\cdot\frac{(0\cdot z+0+1)}{(z+1)^2}\right] - \left[-\exp[-(z+1)\cdot 0]\cdot\frac{(0\cdot z+0+1)}{(z+1)^2} - -\exp[-(z+1)\cdot -\infty]\cdot\frac{(-\infty\cdot z+-\infty+1)}{(z+1)^2}\right]$$

since $$-\exp[-(z+1)\cdot\infty]\cdot\frac{(\infty\cdot z+\infty+1)}{(z+1)^2} = -\exp[-(z+1)\cdot -\infty]\cdot\frac{(-\infty\cdot z+-\infty+1)}{(z+1)^2} = 0$$

the definite integral becomes $$f_z(z) = \left[0-\frac{-1}{(z+1)^2}\right] - \left[\frac{-1}{(z+1)^2}-0\right] = \frac{2}{(z+1)^2}$$

(for $-\infty < z < \infty$)

Note: In equation [7], the left integral supports $0 < z < \infty$ while the right integral supports $-\infty < z < 0$.

(for $0 < z < \infty$)

$$f_z(z) = \frac{1}{(z+1)^2} \qquad [8]$$

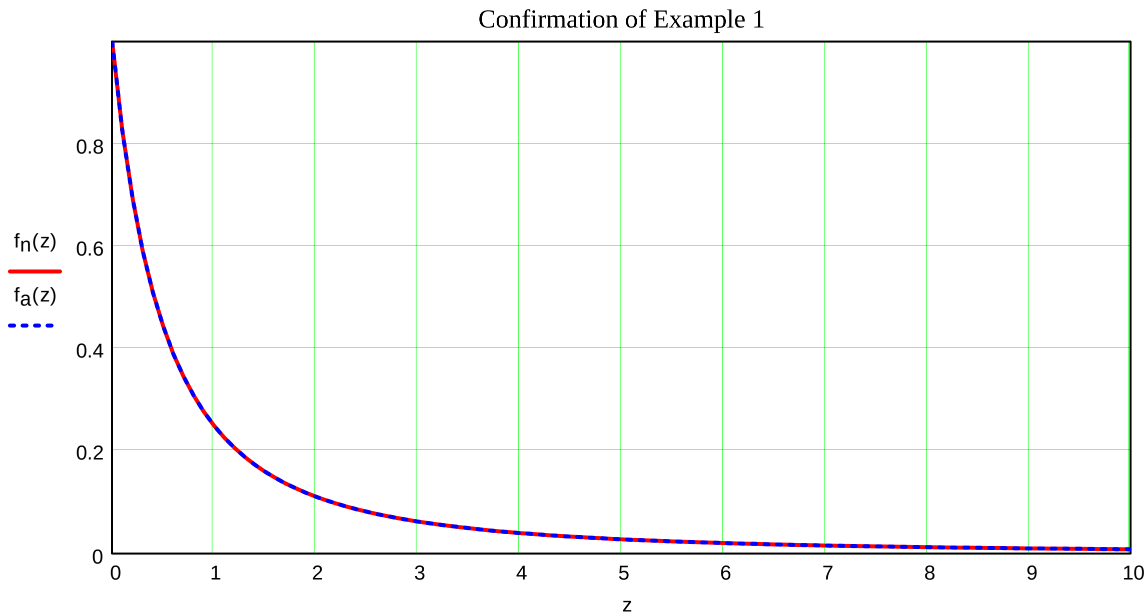
(which is what is given by equation 25 on page 397 of Shooman's book)

A confirmation of the analytical solution is performed below by comparing it to a numerical integration of equation [7] from $0 < z < 10$.

$\lambda_1 := 1$ $\lambda_2 := 1$ $U := 10$ $z := 0, .1 .. 10$

numerical function $$f_n(z) := \left[\int_0^U y\left(\lambda_1 \cdot e^{-\lambda_1 \cdot z \cdot y}\right)\cdot\left(\lambda_2 \cdot e^{-\lambda_2 \cdot y}\right) dy\right]$$

analytical function $$f_a(z) := \frac{1}{(z+1)^2}$$



Case 1 conclusion:

for $0 < z < \infty$ and $\lambda_1 = \lambda_2 = 1$, both curves overlay indicating equation [7] is equivalent to equation [8].

Changing the distribution from f(z) to f(i) where z = 1 + i where i is the interest rate.

Case 2, Example 1 (exponential distributions with equal means) as derived on page 13:

(for 0 < z < ∞)

$$f_z(z) = \frac{1}{(z+1)^2}$$

(which is what is given by equation 25 on page 397 of Shooman's book)

Changing the independent variable to i (interest rate) since z = 1 + i

original limits of z $0 < z < \infty$

substitution since z =1 + i $0 < i + 1 < \infty$

new limits for i $-1 < i < \infty$

remembering

$$F(z) = \int_0^z \frac{1}{(z+1)^2}\,dz$$

making a change in variables and limits:

$$F(i) = \int_{-1}^{i} \frac{1}{[(1+i)+1]^2}\,di$$

where the indefinite integral is:

$$\int \frac{1}{(2+i)^2}\,di = \frac{-1}{(2+i)}$$

calculating the CDF for i using the new limits

$$F(i) = \int_{-1}^{i} \frac{1}{(2+i)^2}\, di = \frac{-1}{[2+(i)]} - \frac{-1}{(2+-1)} = \frac{(1+i)}{(2+i)}$$

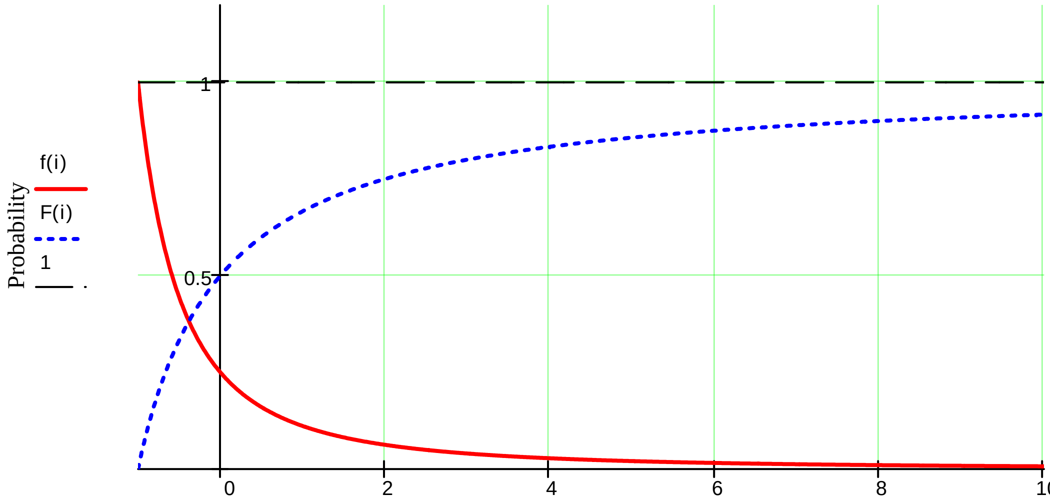
calculating the PDF for i

$$f(i) = \frac{d}{di}\left[\frac{(1+i)}{(2+i)}\right] = \frac{1}{(2+i)} - \frac{(1+i)}{(2+i)^2} \quad \text{or} \quad f(i) := \frac{1}{(2+i)} - \frac{(1+i)}{(2+i)^2}$$

graphing the PDF and CDF

$$f(i) := \frac{1}{(2+i)} - \frac{(1+i)}{(2+i)^2} \quad \text{and} \quad F(i) := \frac{(1+i)}{(2+i)} \quad \text{over the domain:} \quad i := -1, -.99 .. 10$$

Case 2 - Example 1: PDF and CDF vs i



Confirming the CDF = 1:

$$\int_{-1}^{\infty} f(i)\, di = 1$$

Calculating the <u>mean</u> of the quotient of 2 exponential distributions having equal means ($1/\lambda_1 = 1/\lambda_2 = 1$).

definition of the mean (for our PDF):

$$\mu = \int_{-1}^{\infty} i \cdot f(i)\, di$$

computing the indefinite integral (using Mathcad to avoid the tedious integration by parts):

$$\int i \cdot \left[\frac{1}{(2+i)} - \frac{(1+i)}{(2+i)^2} \right] di = \frac{2}{(2+i)} + \ln(2+i)$$

computing the definite integral using the limits: -1 to ∞ :

$$\mu = \frac{2}{(2+\infty)} + \ln(2+\infty) - \left[\frac{2}{[2+(-1)]} + \ln[2+(-1)] \right]$$

$$\mu = 0 + \infty - (-2+0) = 0 + \infty + 2 = \infty$$

$$\mu = \infty$$

Just for kicks, let's do a hand solution and integrate:

$$\int i \cdot \left[\frac{1}{(2+i)} - \frac{(1+i)}{(2+i)^2} \right] di$$

first simplify the integrand:

$$i \cdot \left[\frac{1}{(2+i)} - \frac{(1+i)}{(2+i)^2} \right] = \frac{i}{(2+i)} - \frac{i+i^2}{(2+i)^2} = \frac{\left(2 \cdot i + i^2 - i - i^2\right)}{(2+i)^2} = \frac{i}{\left(4 + 4 \cdot i + i^2\right)} = \frac{i}{(2+i)^2} = i \cdot (2+i)^{-2}$$

thus

$$\int i \cdot \left[\frac{1}{(2+i)} - \frac{(1+i)}{(2+i)^2} \right] di = \int i \cdot (2+i)^{-2} \, di$$

integrating by parts:

$$\int u \, dv = u \cdot v - \int v \, du$$

letting

$$u = i \qquad dv = (2+i)^{-2} di$$

$$du = 1 \cdot di \qquad v = \frac{-1}{(2+i)}$$

substitution

$$\int i \cdot (2+i)^{-2} \, di = i \cdot \frac{-1}{(2+i)} - \int \frac{-1}{(2+i)} \, di$$

thus

$$\int i \cdot (2+i)^{-2} \, di = \frac{-i}{(2+i)} - (-\ln(2+i) + C) = \frac{-i}{(2+i)} + \ln(2+i) - C$$

Solving for C since we know
f(-1) = 0
[the mean is zero when i = -1]:

$$0 = \frac{-(-1)}{[2+(-1)]} + \ln[2+(-1)] - C \qquad C := 1$$

thus

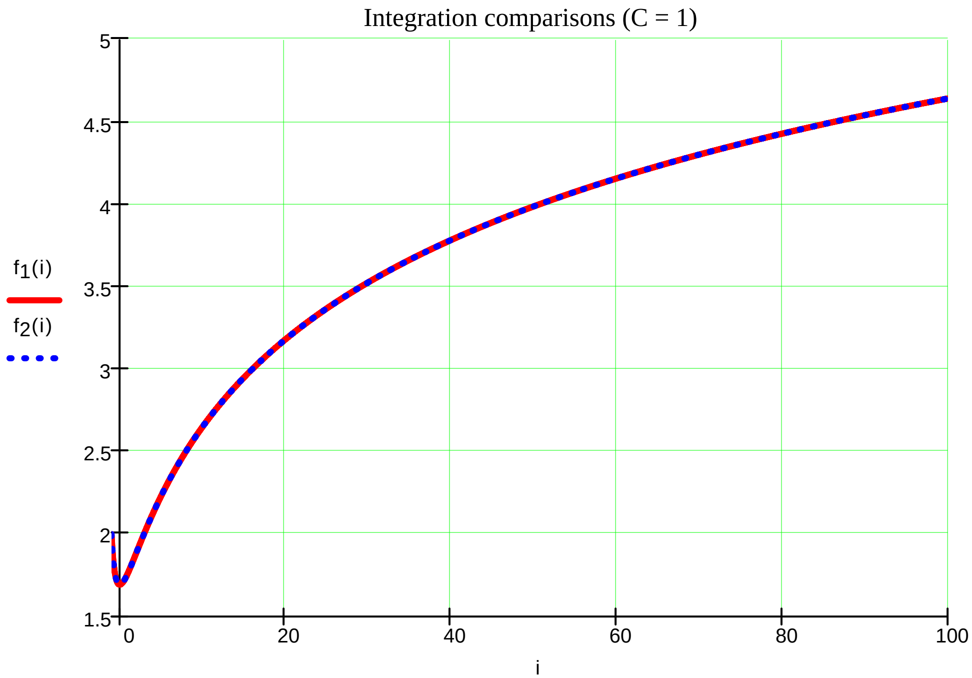
$$\int i \cdot (2+i)^{-2} \, di = \frac{-i}{(2+i)} + \ln(2+i) - 1$$

For confirmation, let's graph both the Mathcad and hand solutions to see if they're equal when: $C := 1$

Mathcad solution $f_1(i) := \frac{2}{(2+i)} + \ln(2+i)$

hand solution $f_2(i) := \frac{-i}{(2+i)} + \ln(2+i) + C$

over the domain $i := -1, -.99 .. 100$



Difference Table for the First 17 Values

$i =$

	0
0	-1
1	-0.99
2	-0.98
3	-0.97
4	-0.96
5	-0.95
6	-0.94
7	-0.93
8	-0.92
9	-0.91
10	-0.9
11	-0.89
12	-0.88
13	-0.87
14	-0.86
15	-0.85
16	-0.84
17	-0.83

$f_1(i) - f_2(i) =$

	0
0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0
10	0
11	0
12	0
13	0
14	0
15	0
16	0
17	0

Example 2: **Letting** $\lambda_1 \neq \lambda_2$ in $f_z(z) = \int_0^{\infty} y\left(\lambda_1 \cdot e^{-\lambda_1 \cdot zy}\right)\cdot\left(\lambda_2 \cdot e^{-\lambda_2 \cdot y}\right) dy - \int_{-\infty}^{0} y \cdot\left(\lambda_1 \cdot e^{-\lambda_1 \cdot zy}\right)\cdot\left(\lambda_2 \cdot e^{-\lambda_2 \cdot y}\right) dy$

and remembering from equation [7a]: $\int y \cdot \lambda_1 \cdot e^{-\lambda_1 \cdot z \cdot y} \cdot \lambda_2 \cdot e^{-\lambda_2 \cdot y} dy = \frac{\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot y \cdot \exp\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot y\right] - \exp\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot y\right]\right]}{\left(-\lambda_1 \cdot z - \lambda_2\right)^2}\cdot \lambda_1 \cdot \lambda_2$

Applying the limits of integration to the first integral in $f_z(z)$

$$\int_0^{\infty} y\left(\lambda_1 \cdot e^{-\lambda_1 \cdot zy}\right)\cdot\left(\lambda_2 \cdot e^{-\lambda_2 \cdot y}\right) dy = \frac{\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot \infty \cdot \exp\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot \infty\right] - \exp\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot \infty\right]\right]}{\left(-\lambda_1 \cdot z - \lambda_2\right)^2}\cdot \lambda_1 \cdot \lambda_2$$

$$- \left[\frac{\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot 0 \cdot \exp\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot 0\right] - \exp\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot 0\right]\right]}{\left(-\lambda_1 \cdot z - \lambda_2\right)^2}\cdot \lambda_1 \cdot \lambda_2\right]$$

since: $\frac{\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot \infty \cdot \exp\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot \infty\right] - \exp\left[\left(-\lambda_1 \cdot z - \lambda_2\right)\cdot \infty\right]\right]}{\left(-\lambda_1 \cdot z - \lambda_2\right)^2}\cdot \lambda_1 \cdot \lambda_2 = 0$

so... $\int_0^{\infty} y\left(\lambda_1 \cdot e^{-\lambda_1 \cdot zy}\right)\cdot\left(\lambda_2 \cdot e^{-\lambda_2 \cdot y}\right) dy = 0 + \lambda_1 \cdot \frac{\lambda_2}{\left(\lambda_1 \cdot z + \lambda_2\right)^2}$

Applying the limits of integration to the second integral in $f_z(z)$

$$\int_{-\infty}^{0} y\cdot\left(\lambda_1\cdot e^{-\lambda_1\cdot zy}\right)\cdot\left(\lambda_2\cdot e^{-\lambda_2\cdot y}\right)dy = \frac{\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot 0\cdot \exp\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot 0\right]-\exp\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot 0\right]\right]}{\left(-\lambda_1\cdot z-\lambda_2\right)^2}\cdot\lambda_1\cdot\lambda_2$$

$$-\left[\frac{\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot -\infty\cdot \exp\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot -\infty\right]-\exp\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot -\infty\right]\right]}{\left(-\lambda_1\cdot z-\lambda_2\right)^2}\cdot\lambda_1\cdot\lambda_2\right]$$

since:

$$\frac{\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot -\infty\cdot \exp\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot -\infty\right]-\exp\left[\left(-\lambda_1\cdot z-\lambda_2\right)\cdot -\infty\right]\right]}{\left(-\lambda_1\cdot z-\lambda_2\right)^2}\cdot\lambda_1\cdot\lambda_2 = 0$$

so...

$$\int_{-\infty}^{0} y\cdot\left(\lambda_1\cdot e^{-\lambda_1\cdot zy}\right)\cdot\left(\lambda_2\cdot e^{-\lambda_2\cdot y}\right)dy = -\lambda_1\cdot\frac{\lambda_2}{\left(\lambda_1\cdot z+\lambda_2\right)^2}+0$$

The PDF of the quotient of two random variables derived from different exponential functions from $-\infty < z < \infty$ is:

$$f_z(z) = \lambda_1\cdot\frac{\lambda_2}{\left(\lambda_1\cdot z+\lambda_2\right)^2}-\left[-\lambda_1\cdot\frac{\lambda_2}{\left(\lambda_1\cdot z+\lambda_2\right)^2}\right] = 2\cdot\lambda_1\cdot\frac{\lambda_2}{\left(\lambda_1\cdot z+\lambda_2\right)^2}$$

($-\infty < z < \infty$)

The PDF of the quotient of two random variables derived from exponential functions having different means (from $0 < z < \infty$) is:

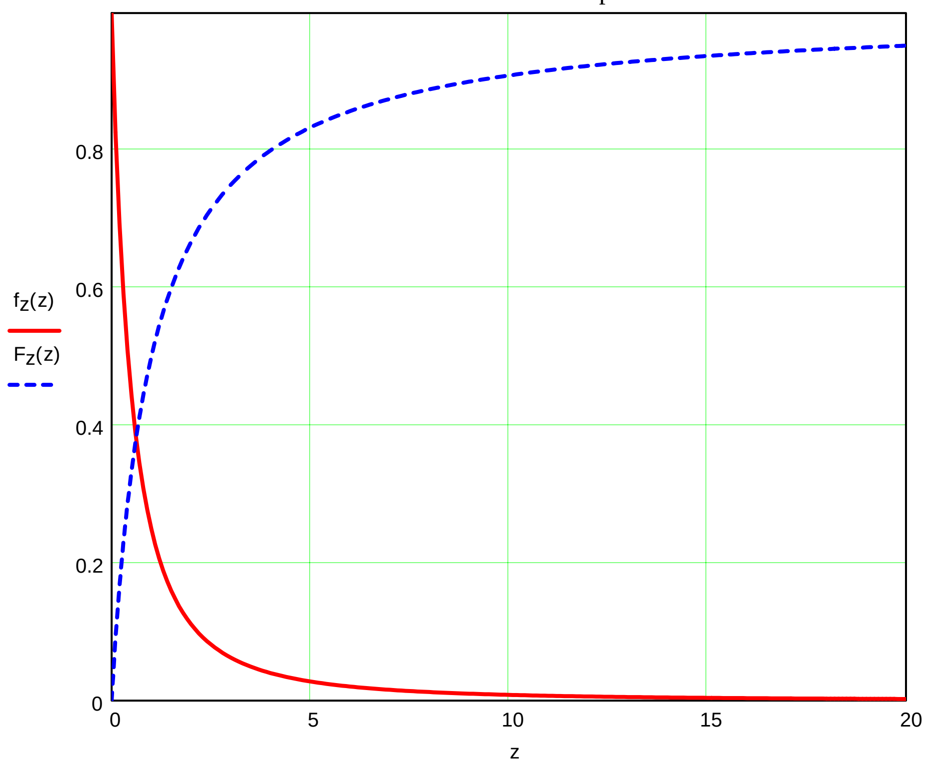
$$f_z(z) = \lambda_1\cdot\frac{\lambda_2}{\left(\lambda_1\cdot z+\lambda_2\right)^2}$$

($0 < z < \infty$)

Checking the PDF and CDF for example 2:

$\lambda_1 := 1$ $\lambda_2 := 1$ $z := 0, .1 .. 20$ $f_z(z) := \lambda_1 \cdot \frac{\lambda_2}{(\lambda_1 \cdot z + \lambda_2)^2}$ $F_z(z) := \int_0^z \lambda_1 \cdot \frac{\lambda_2}{(\lambda_1 \cdot z + \lambda_2)^2} dz$

PDF and CDF - Example 2



Checking the CDF numerically with a large number to ensure it equals 1.

$F_z(10000) = 1$

or, another way

$$\lim_{z \to \infty} \left[1 - \lambda_1 \cdot \frac{\lambda_2}{(\lambda_1 \cdot z + \lambda_2)^2} \right] \to 1$$

Changing the distribution from f(z) to f(i) where z = 1 + i where i is the interest rate.

Case 2, Example 2 (exponential distributions with unequal means) was derived on pages 20 - 21:

(for 0 < z < ∞)

$$f_z(z) = \lambda_1 \cdot \frac{\lambda_2}{(\lambda_1 \cdot z + \lambda_2)^2}$$

Letting (for example): $\lambda_1 := 1$ $\lambda_2 := 2$

Changing the independent variable to i (interest rate) since z = 1 + i

original limits of z $0 < z < \infty$

substitution since z =1 + i $0 < i + 1 < \infty$

new limits for i $-1 < i < \infty$

remembering

$$F(z) = \int_0^z \lambda_1 \cdot \frac{\lambda_2}{(\lambda_1 \cdot z + \lambda_2)^2}\, dz$$

making a change in variables and limits:

$$F(i) = \int_{-1}^{i} \lambda_1 \cdot \frac{\lambda_2}{\left[\lambda_1 \cdot (1 + i) + \lambda_2\right]^2}\, di$$

where the indefinite integral is:

$$\int \lambda_1 \cdot \frac{\lambda_2}{\left[\lambda_1 \cdot (1 + i) + \lambda_2\right]^2}\, di = \frac{-1}{(\lambda_1 \cdot i + \lambda_1 + \lambda_2)} \cdot \lambda_2$$

calculating the CDF for i using the new limits

$$F(i) = \int_{-1}^{i} \lambda_1 \cdot \frac{\lambda_2}{\left[\lambda_1 \cdot (1+i) + \lambda_2\right]^2}\, di = \frac{-1}{\left(\lambda_1 \cdot i + \lambda_1 + \lambda_2\right)} \cdot \lambda_2 - \frac{-1}{\left[\lambda_1 \cdot (-1) + \lambda_1 + \lambda_2\right]} \cdot \lambda_2$$

which simplifies to:

$$F(i) := \lambda_1 \cdot \frac{(1+i)}{\left(\lambda_1 \cdot i + \lambda_1 + \lambda_2\right)} \qquad \text{(CDF)}$$

calculating the PDF for i

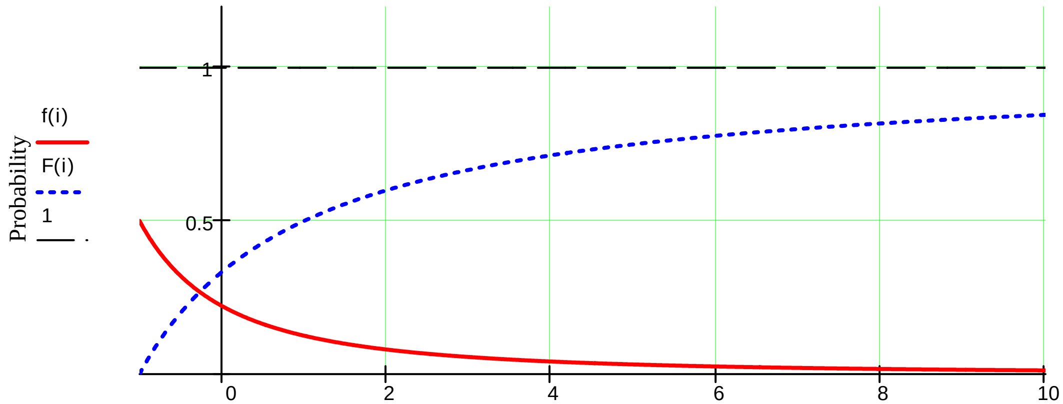
$$f(i) = \frac{d}{di}\left[\lambda_1 \cdot \frac{(1+i)}{\left(\lambda_1 \cdot i + \lambda_1 + \lambda_2\right)}\right] = \frac{\lambda_1}{\left(\lambda_1 \cdot i + \lambda_1 + \lambda_2\right)} - \lambda_1^{\,2} \cdot \frac{(1+i)}{\left(\lambda_1 \cdot i + \lambda_1 + \lambda_2\right)^2}$$

so

$$f(i) := \left[\frac{\lambda_1}{\left(\lambda_1 \cdot i + \lambda_1 + \lambda_2\right)} - \lambda_1^{\,2} \cdot \frac{(1+i)}{\left(\lambda_1 \cdot i + \lambda_1 + \lambda_2\right)^2}\right] \qquad \text{(PDF)}$$

graphing the PDF and CDF (remembering $\lambda_1 = 1$ and $\lambda_2 = 2$) over the domain: $i := -1, -.99 .. 10$

Case 2 - Unequal means PDF and CDF vs i



Confirming the CDF = 1:

$$\int_{-1}^{\infty} f(i)\, di = 1$$

Calculating the **mean** of the quotient of 2 exponential distributions having **unequal** means.

definition of the mean (for our PDF):

$$\mu = \int_{-1}^{\infty} i \cdot f(i)\, di$$

Next, computing the indefinite integral

$$\int i \cdot \left[\frac{\lambda_1}{(\lambda_1 \cdot i + \lambda_1 + \lambda_2)} - \lambda_1^{\,2} \cdot \frac{(1+i)}{(\lambda_1 \cdot i + \lambda_1 + \lambda_2)^2} \right] di = \frac{\lambda_2}{(\lambda_1 \cdot i + \lambda_1 + \lambda_2)} + \frac{1}{\lambda_1 \cdot (\lambda_1 \cdot i + \lambda_1 + \lambda_2)} \cdot \lambda_2^{\,2} + \frac{1}{\lambda_1} \cdot \lambda_2 \cdot \ln(\lambda_1 \cdot i + \lambda_1 + \lambda_2)$$

computing the definite integral using the limits: -1 to ∞ :

$$\mu = \int_{-1}^{\infty} \frac{\lambda_2}{(\lambda_1 \cdot i + \lambda_1 + \lambda_2)} + \frac{1}{\lambda_1 \cdot (\lambda_1 \cdot i + \lambda_1 + \lambda_2)} \cdot \lambda_2^{\,2} + \frac{1}{\lambda_1} \cdot \lambda_2 \cdot \ln(\lambda_1 \cdot i + \lambda_1 + \lambda_2)\, di$$

or

$$\mu = \frac{\lambda_2}{(\lambda_1 \cdot \infty + \lambda_1 + \lambda_2)} + \frac{1}{\lambda_1 \cdot (\lambda_1 \cdot \infty + \lambda_1 + \lambda_2)} \cdot \lambda_2^{\,2} + \frac{1}{\lambda_1} \cdot \lambda_2 \cdot \ln(\lambda_1 \cdot \infty + \lambda_1 + \lambda_2)$$
$$- \left[\frac{\lambda_2}{[\lambda_1 \cdot (-1) + \lambda_1 + \lambda_2]} + \frac{1}{\lambda_1 \cdot [\lambda_1 \cdot (-1) + \lambda_1 + \lambda_2]} \cdot \lambda_2^{\,2} + \frac{1}{\lambda_1} \cdot \lambda_2 \cdot \ln[\lambda_1 \cdot (-1) + \lambda_1 + \lambda_2] \right]$$

thus

$$\mu = \infty - (3 + 2 \cdot \ln(2)) = \infty$$

Taking the limits of both parts of the definite integral just to be sure:

$$\lim_{i \to \infty} \left[\frac{\lambda_2}{(\lambda_1 \cdot i + \lambda_1 + \lambda_2)} + \frac{1}{\lambda_1 \cdot (\lambda_1 \cdot i + \lambda_1 + \lambda_2)} \cdot \lambda_2^{\,2} + \frac{1}{\lambda_1} \cdot \lambda_2 \cdot \ln(\lambda_1 \cdot i + \lambda_1 + \lambda_2) \right] \to \infty$$

$$\lim_{i \to -1} \left[\frac{\lambda_2}{(\lambda_1 \cdot i + \lambda_1 + \lambda_2)} + \frac{1}{\lambda_1 \cdot (\lambda_1 \cdot i + \lambda_1 + \lambda_2)} \cdot \lambda_2^{\,2} + \frac{1}{\lambda_1} \cdot \lambda_2 \cdot \ln(\lambda_1 \cdot i + \lambda_1 + \lambda_2) \right] \to 3 + 2 \cdot \ln(2)$$

Summary Page

The Investment Problem: $0 = A_0 + \frac{A_1}{1+i}$

	A_o	A_1	Remarks	PDF	Mean	CDF
Case 1	constant = k	exponential (lambda)		$f(i) := \lvert k \rvert \cdot \lambda \cdot \exp[\lambda \cdot k \cdot (1+i)]$	$\mu = 0$	$F(i) := \lvert k \rvert \cdot \frac{[\exp[\lambda \cdot k \cdot (1+i)] - 1]}{k}$
Case 2						
Example1	exponential (lambda 1)	exponential (lambda 2)	both means = 1	$f(i) := \frac{1}{(2+i)} - \frac{(1+i)}{(2+i)^2}$	$\mu = \infty$	$F(i) := \frac{(1+i)}{(2+i)}$
Example 2	exponential (lambda 1)	exponential (lambda 2)	unequal means	$f(i) := \left[\frac{\lambda_1}{(\lambda_1 \cdot i + \lambda_1 + \lambda_2)} - \lambda_1^2 \cdot \frac{(1+i)}{(\lambda_1 \cdot i + \lambda_1 + \lambda_2)^2}\right]$	$\mu = \infty$	$F(i) := \lambda_1 \cdot \frac{(1+i)}{(\lambda_1 \cdot i + \lambda_1 + \lambda_2)}$