Probability Distribution Function of the Internal Rate of Return in One and Two Period Engineering Economy Problems with Random Cash Flows

by

Gökçe Palak

A Thesis

Submitted to
the Department of Engineering
Colorado State University-Pueblo

In partial fulfillment of the requirements
For the degree of
Master of Science

Completed on April 29, 2009

Abstract

Gökçe Palak for the degree of **Master of Science** in **Industrial and Systems Engineering** presented on April 29, 2009.

Title:

Probability Distribution Function of the Internal Rate of Return in One and Two Period Engineering Economy Problems with Random Cash Flows

Abstract approved: ____________________

Hüseyin Sarper, Ph.D.,P.E.

This thesis finds the closed form probability distribution expressions of the internal rate of return for certain one and two period stochastic engineering economy problems.
In each type of the problem, the roots of the internal rate of return are derived initially. The probability distribution of the internal rate of return is then found for different combinations of random cash flows. These combinations are constructed by increasing the number of the random cash flows. In one period problem, two different combinations are considered where all cash flows and the cash flow of period one are random variables. In two period problem, two cases are constructed where the cash flow of period one or period two are random variables individually. The random cash flows are considered to be either uniform or exponential variables. For each combination considered, the probability distribution of the internal rate of return is derived with the distribution function method and transformation of variables method if possible. It is shown that the same solutions could be derived with either method. The cumulative distribution functions, the expected values and the variances of these problems are also established. The analytical results are verified with the simulation results for each type of problems. Cumulative distribution function is mainly used in order to interpret the probabilistic internal rate of return; therefore numerical examples are also illustrated.

Certificate of Acceptance

This thesis
Presented impartial fulfillment of
the requirements for the degree of

Master of Science

APPROVED

Dr. Hüseyin Sarper, P.E. — Date
Professor of Engineering and Committee Chair

Dr. Jane M. Fraser — Date
Professor and Chair of Engineering and Committee Member

Dr. Paul R. Chacon — Date
Professor and Chair of Mathematics and Physics and Committee Member

Mr. Mark E. Heinen — Date
Mathematics Teacher, Cañon City High School and Committee Member

Master’s Candidate — Gökçe Palak
Date of thesis presentation — April 29, 2009.

Acknowledgements

“The actual science of logic is conversant at present only with things either certain, impossible, or entirely doubtful, none of which (fortunately) we have to reason on. Therefore the true logic for this world is the calculus of probabilities, which takes account of the magnitude of the probability which is, or ought to be, in a reasonable man's mind.”

James Clark Maxwell

Table of Contents

Table of Figures

List of Tables

Chapter 1 – Introduction

The internal rate of return (IRR) has long been used as one of the several major indices in determining the desirability of a given investment such as a new project or an incremental investment between two projects. It is defined as the interest paid or earned on the unrecovered balance such that the initial principal and interest are completely recovered with the final payment. This method solves for the interest rate that equates the equivalent worth of cash inflows (receipts or savings) to the equivalent worth of the cash outflows (expenditures, including investments) [8].

For a deterministic single alternative, the formulation of $IRR = i'$ is defined as:

$$\sum_{k=0}^{N} R_k(P|F, i'\%, k) = \sum_{k=0}^{N} E_k(P|F, i'\%, k) \qquad (1)$$

where R_k net receipts or savings are for kth year, E_k are net expenditures including investments for kth year and N stands for project life. The alternative is acceptable when $i' \geq MARR$ (minimum attractive rate of return). The reader is referred to Appendix A for alternate views on the IRR definition by Renwick in 1971 [37].

This thesis especially deals with the stochastic engineering economy problems. In the deterministic case, the cash flows are assumed to be known or deterministic amounts. This assumption may not be valid in some real life applications. If the cash flows are random variables, the IRR will have a probability density function (PDF). The objective of this thesis is to find a closed form PDF expression for the IRR of certain one period and two period cash flow engineering economy problems.

If the following typical features are random, projects become more real and, naturally, more stochastic:

1. Project duration,
2. The timing between the successive net cash flows,
3. Cash outflows (including the initial investment),
4. Revenues,
5. Applicable tax and/or MARR rates,
6. Salvage value.

All these features may exhibit no, partial, or significant dependence expressed via correlation coefficient matrix and/or joint PDFs based on historical evidence and/or experience. This thesis does not consider any correlation among the cash flows.

Many studies with stochastic cash flows have focused on the PDF of the Net Present Value (NPV) and various versions of the cost/benefit ratios, especially when normality assumption holds. Major advanced texts ([33], [37], [42], [44]) provide excellent procedures, including simulation, to determine the PDF or at least the parameters of the random NPV using independent or correlated cash flows. The PDF or even parameters of the IRR is notably absent in most texts in economic analysis. The randomness of the IRR has been considered since the 60's, but never at the popularity level of the randomness of the NPV.

The PDF of the IRR is a meaningful tool because the decision maker needs to know the probability of the IRR exceeding the MARR. However, Ekern [10] disagrees with the usage of the probabilistic IRR when the type of the project is uncertain or not considered at the time of the analysis. The type of the investment projects should also be discussed in the determination of IRR criterion. Investment projects can be classified as simple vs. nonsimple and pure vs. mixed investments. Simple investments are the ones where there is only one sign change in the cash flow. These investments have the only investment at the beginning of the project and the following cash flows are the revenues. Nonsimple investments have cash outflows that can be in any period of the project. Pure investments are defined as the ones which have the project balances at the end of each period either negative or zero. To the contrary, pure financing projects should have the project balances always positive at the end of each period. Mixed investments are combination of positive and negative project balances in different periods, meaning that the company can borrow or invest money [33].

According to Ekern [10], when the type of the project is not certain in terms of pure investment or pure financing, the PDF of IRR can lead to confusion when the same projects are compared in terms of their NPVs. The conditions for accepting or rejecting a project according to their IRRs change when the type of the project is either pure investment or pure financing. His paper [10] discusses with an example that the IRRs do not yield the same results with the NPVs when the type of the project is not specified. In addition to the uncertain type of investment, he also states that the probabilistic IRR can cause serious problems. It may have some scaling problems which would be inconsistent with the probabilistic NPV. The distributions of the cash flows may also not be statistically appropriate for the project.

Derivation of the PDF for the IRR makes sense only if the well-known re-investment at the IRR assumption is valid and any multiple root related complications are adequately resolved. Otherwise, the IRR and its PDF have much less financial meaning. For simple projects, the PDF of IRR would be an extremely effective decision tool. In this study, the projects considered are assumed to be pure and simple investments to avoid the difficulty of IRR applications due to cash flow sign changes and multiple roots. This thesis does not address such issues.

This thesis seeks to extend the one period problem that was considered by Heinen [21]. In the context of this study, the probabilistic IRR problem is solved for both one and two period problems. A fully solved case includes a proper PDF and the moments (first and second) of that PDF.

Case 1: The One Period Problem

This problem deals with one period long project with an initial time zero investment of A_0 and positive net cash inflow of A_1 as shown in Figure 1 below. Let *i* be the random IRR variable. If at least A_0 is a random cash flow with its own PDF, then *i* will have a PDF that is a function of the PDFs for A_0 and A_1. Complete PDF information for A_0 and A_1 may not be available, but such availability is assumed. A_1 can be less than A_0, but this does not present a problem. The IRR is normally defined in the range of -1 or -100% to infinity.

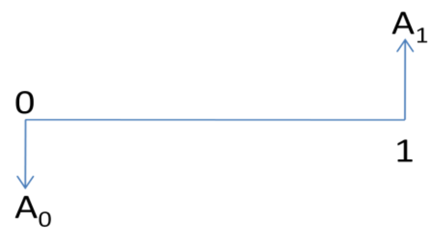

Figure 1 . One Period Problem Cash Flow Diagram

Writing the IRR equation,

$$-A_0 + \frac{A_1}{(1+i)} = 0 \tag{2}$$

$$i = \frac{A_1}{A_0} - 1 \tag{3}$$

A_0 is a positive value by default and its negativity is maintained by the minus sign in the formulation. Two cases are considered in order to find the distribution of the IRR:

The first case is when A_0 is a positive constant and A_1 is a random variable. Then *i* will only have a distribution that depends on the distribution of A_1. In the second case both of the cash flows are assumed to be random variables. If it is before the investment time – time t-1, t-2, or even an earlier period – the A_0 (or investment) may well appear random. So, A_0 can be also considered as a random variable when future investments are under study. A_1 should be random when A_0 is random. If both A_0 and A_1 are stochastic cash flows, the PDF of *i* is dependent on the quotient of these two PDFs. Quotient of random variables is a difficult process to model into a PDF. The reader is referred to references [14], [15], [16], [24] for additional information.

In both of the cases the distribution of the IRR can be derived by first deriving either the cumulative density function (CDF) or using transformation of the random variables method.

The one period problem with exponentially distributed cash flows has been considered by Heinen [21]. His study is the inspiration and the motivation for this thesis. One period problem is further studied in this thesis with uniform cash flows in this study to complement Heinen's study.

Heinen [21] derives the PDF of IRR for one period problem with exponential cash flows. The PDF, the CDF and the expected values are derived in his study. Heinen does not calculate the variances. He confirms that the CDF of each case is equal to one. The analytical results are applied in examples for verification. His results are reproduced in Table 1 below.

In the first case, Heinen takes the initial cash flow as a constant (k) and considers the cash flow of period one (x) as an exponential random variable. 'x' is assumed to be exponentially distributed with a rate of λ and a mean of μ. For exponential distributions, λ refers to 1/μ.

'k' is assumed to be negative by default. Using the fundamental theorem in Papoulis' text [35] , he derives the CDF and the PDF of the IRR. The expected value is found to be 0 for this case.

In the second case, both the initial cash flow and the cash flow of period one are assumed to be exponential random variables. The root of the IRR is found to follow the distribution of the quotient of these two random variables. An analysis of the quotient of two random variables is first discussed in his study. Then two different examples are explored for when cash flows have equal means and unequal means. The PDF and the CDF formulas are derived and the expected values for both cases are found to be equal to infinity.

Table 1 . One Period Problem Results with Exponential Cash Flows [21]

CASES	REMARKS	PDF	CDF	EXPECTED VALUE
A_0 constant, A_1 ~ exp (λ)		$f(i) = \|k\|\lambda e^{\lambda k(1+i)}$	$F(i) = \|k\|\frac{e^{\lambda k(1+i)}-1}{k}$	0
A_0 ~ exp (λ_1), A_1 ~ exp (λ_2)	$\frac{1}{\lambda_1}=\frac{1}{\lambda_1}=1$	$f(i)=\frac{1}{2+i}-\frac{1+i}{(2+i)^2}$	$F(i)=\frac{1+i}{2+i}$	∞
A_0 ~ exp (λ_1), A_1 ~ exp (λ_2)	$\frac{1}{\lambda_1}\neq\frac{1}{\lambda_1}$	$f(i)=\frac{\lambda_1}{\lambda_1 i+\lambda_1+\lambda_2}-\lambda_1^{\,2}\frac{1+i}{(\lambda_1 i+\lambda_1+\lambda_2)^2}$	$F(i)=\lambda_1\frac{1+i}{(\lambda_1 i+\lambda_1+\lambda_2)}$	∞

Monte-Carlo simulation can always be used to determine the PDF of the IRR for each given PDF set for A_0 and A_1. The @RISK [32] software can be easily used even if the two cash flows are also correlated. Simulation results can be fitted to one of the PDFs using an available statistical package. The @RISK [32] software can also do this for the simulation data it creates. The functions obtained by the analytical solution are therefore compared to the Monte Carlo simulation results for specific problems for verification. It is reasonable to use simulation to verify an analytical result. An all purpose analytical formula, the PDF in this thesis, is a superior solution to simulation in general.

Case 2: The Two Period Problem

This problem considers a two period long project with an initial time zero investment of A_0 and positive net cash inflows of A_1 and A_2. Again let *i* be the random IRR variable.

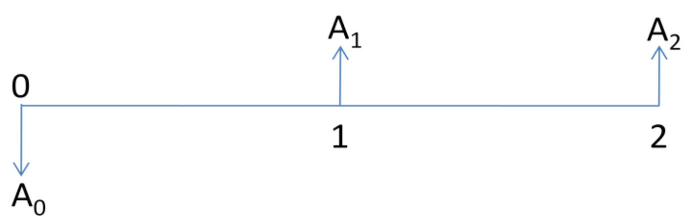

Figure 2 . Two Period Problem Cash Flow Diagram

$$-A_0 + \frac{A_1}{(1+i)} + \frac{A_1}{(1+i)^2} = 0 \qquad (4)$$

Let $p = 1 + i$

$$-A_0 + \frac{A_1}{p} + \frac{A_2}{p^2} = 0 \qquad \text{and} \qquad -A_0p^2 + A_1p + A_2 = 0$$

Using the classic quadratic function format of $ap^2 + bp + c = 0$ where $a = -A_0,\ b = A_1$, $c = A_2$, the roots are found as follows:

$$\Delta = \sqrt{b^2 - 4ac} = \sqrt{{A_1}^2 + 4A_0A_2}$$

$$p_{1,2} = \frac{-A_1 \pm \sqrt{{A_1}^2 + 4A_0A_2}}{-2A_0}$$

$$i_{1,2} = \frac{-A_1 \pm \sqrt{{A_1}^2 + 4A_0A_2}}{-2A_0} - 1 = \frac{-A_1 + 2A_0 \pm \sqrt{{A_1}^2 + 4A_0A_2}}{-2A_0} \qquad (5)$$

Equation 5 represents the problem of finding the PDF of the roots of a polynomial equation. This problem is first reported by Erdos and Turan [12] for a general polynomial in 1950. Later, Hamblen [20] presents a solution for the quadratic case. These two publications ([12], [20]) appear highly relevant but this M.S. candidate was unable to understand them fully.

Two different cases are considered in this problem. In the first case, the A_2 is the only random cash flow. Then the density function of *i* or the IRR depends only on the square root transformation of the distribution of A_2. It can also be solved by the distribution function method. In the second case, A_1 is the only random variable. This case was only solved by the distribution function method. Transformation of random variables method gets highly complicated as sum and square root transformations are not independent of each other. Term A_0 is always kept as deterministic due to the limited time work for this thesis.

In case when the PDFs cannot be found, mean and standard deviation of the IRR may be found by the Taylor series expansion or the Mellin transforms. This is a challenging problem even if complete independence is assumed among the three cash flows. For this research, the random cash flows are assumed to be independent random variables and inclusion of correlation between random variables is suggested for further research.

For the two period problem, the cases described above are examined by using two different distributions for the random cash flows (exponential and uniform). Another possible classification is changing the signs of the cash flows, and naturally, the order of the revenues and the investments. By having only one sign change in the two period IRR problem, additional complications in the interpretation of the IRR is avoided. Much celebrated two period oil – pump problem with -, +, - signs [4] is difficult even in its deterministic format. Its PDF is probably not only unattainable for the stochastic version, but meaningless too.

Chapter 2 – Literature Review

A very thorough literature review has been conducted on the PDF of IRR. The concept of randomness of the IRR was first introduced in early 60's by Hillier [22]. It appears that this idea has been popular in the 70's. Renwick's [37] text is an exception to many standard texts in economic analysis and finance. His text [37] provides a discussion for the idea of having a PDF for the IRR. Another unique reference belongs to Fairley and Jacoby [13] who are the only authors who provide an actual PDF expression and its plot and use it in assessing the worth of a hydroelectric power investment by the World Bank in Argentine in 1975. The authors [13] consider correlation among the cash flows and introduce the term "random polynomial". As in Hillier's [22] and several other's papers ([13], [27]), the authors also suggest normal approximations for the PDF of the IRR.

Hillier's article [22] deals with the derivation of the probability distributions of the NPV, annual cost and the IRR in order to evaluate the risk of an investment. He classifies simplified procedures and theoretical procedures in the context of the previous studies. He states that the most of the previous simplified procedures that include probability reduce the estimates of each cash flow to a single expected value and not mention the variance. This situation ignores the fact that variance is also a decision factor as one would choose small variance when the expected rate of returns are the same. Theoretical procedures suggested previously include sensitivity analysis, determination of the expected value of the utility and the selection of the investments which have the expected rate of return greater than the cost of capital. Hillier's [22] procedure can be classified in between the simplified and theoretical procedures. He finds the PDF of the IRR beyond the simplified procedures and therefore allows the evaluation of the expected utility as in theoretical procedures.

In the formulation of Hillier's [22] problem all X_j's , the cash flows during j^{th} year are assumed to be normal. Although this assumption cannot be completely justified, it is a valid idea to assume a symmetrical distribution which would seem like normal distribution. Even if the cash flows are not definitely normal random variables, their sum of the present worth would yield an approximately normal distribution due to Central Limit Theorem.

Three cases are considered for the cash flows in this problem as mutually independent, completely correlated and a combination of both. The present worth, P, is given by equation 6.

$$P = \sum_{j=0}^{n} \left[\frac{X_j}{(1+i)^j} \right] \tag{6}$$

If the cash flows are mutually independent, then the expected value and the variance of the present worth would be as in equations 7 and 8 respectively:

$$\mu_p = \sum_{j=0}^{n} \left[\frac{\mu_j}{(1+i)^j} \right] \tag{7}$$

$$\sigma_p^2 = \sum_{j=0}^{n} \left[\frac{\sigma_j^2}{(1+i)^{2j}} \right] \tag{8}$$

If the cash flows are perfectly correlated with a correlation coefficient C, then the mean and the standard deviation of the NPV would be as in the equations 9 and 10.

$$\mu_p = \sum_{j=0}^{n} \left[\frac{\mu_j + C\sigma_j}{(1+i)^j}\right] \tag{9}$$

$$\sigma_p = \sum_{j=0}^{n} \left[\frac{\sigma_j}{(1+i)^j}\right] \tag{10}$$

If the model includes both of the assumptions above then each cash flow would have correlated and independent parts. The cash flows are redefined in equation 11 where Y_j's are independent and Z_j's are perfectly correlated.

$$X_j = Y_j + Z_j^{(1)} + Z_j^{(2)} + Z_j^{(3)} + \cdots + Z_j^{(m)} \tag{11}$$

For this case, the expected value and the variance of the present worth equation will become:

$$\mu_p = \sum_{j=0}^{n} \left[\frac{\mu_j}{(1+i)^j}\right] = \sum_{j=0}^{n} \left[\frac{E(Y_j) + \sum_{k=1}^{m} E\left(Z_j^{(k)}\right)}{(1+i)^j}\right] \tag{12}$$

$${\sigma_p}^2 = \sum_{j=0}^{n} \left[\frac{Var(Y_j)}{(1+i)^{2j}}\right] + \sum_{k=1}^{m} \left(\sum_{j=0}^{n} \left[\frac{\sqrt{Var\left(Z_j^{(k)}\right)}}{(1+i)^j}\right]\right)^2 \tag{13}$$

The IRR or R, is the value of the interest rate for which the present worth equals zero. The investments with the highest R are likely to be chosen. The procedure to find the PDF of R involves *finding the PDF of NPV for various values if i in order to find the cumulative distribution of R and then, if desired, deriving the probability density function of R from the cumulative distribution function* [22].

Renwick's text [37] summarizes Hillier's method and shows how the CDF is constructed. When i is equal to the R, then the P is zero, as in equation 4. If i is greater than R, the P is found to be negative. Similarly when i is less than R, the P is positive. When the cash flows are random variables, the area where the P is less than the assumed rate of discount gives the CDF of IRR. Therefore the probability that R is less than i is the same as the probability that P is less than zero as stated in equation 14.

$$Prob(R < i) = Prob(P < 0|i) \tag{14}$$

The CDF is found by repeating this calculation for various values of i. Although it is enough to evaluate an investment by analyzing the CDF, it is also a good practice to visualize the PDF by taking the first derivative of the CDF.

This procedure is illustrated with the following example directly taken from Hillier's article [22]:

The XYZ Company is primarily engaged in the manufacture of cameras. They will soon be discontinuing the production of one of their older models, and they are now investigating what should be done with the extra productive capacity that will consequently become available. Two attractive alternatives appear to be available. The first alternative is to expand the production of model A, one of their latest and most popular models. This model was initially marketed last year, and its successful reception plus favorable marketing research indicates that there is and will continue to be a market for this extra production. The second alternative is to initiate the production of model B. Model B would involve a number of revolutionary changes which the research department has developed. While no comparable model is now on the market, rumors in the industry indicate that a number of other companies might now have similar models on their drawing boards. Marketing research indicates an exciting but uncertain potential for such a model. Uncertainty regarding the reliability of the proposed new devices, lack of production experience on such a model, and the possibility that the market might be vigorously invaded by competing models at any time, all add to the risk involved in this alternative. In short, the decision is between the safe, conservative investment in model A, or the risky but promising investment in model B. It is felt that both of these models will be marketable for the next five years. Due to a lack of investment funds and productive capacity, it has been decided that only one of these alternatives can be selected. It is assumed that the production of model B would not affect the market for the presently scheduled production of model A.

Detailed studies have been made regarding the after tax cash flow consequences of the two alternatives. The analysis of the investment required in model A indicates that considerable *new equipment, tooling, and modification of existing production processes will be needed. It was estimated that the difference in the immediate cash flow because of the investment in model A would be (-$400,000). However, it is recognized that this estimate is only approximate, so that it is appropriate to estimate the standard deviation for this cash flow. Recalling that the probability is 0.6827, 0.9545, and 0.9973, respectively, that the actual cash flow will be within one, two, and three standard deviations, respectively, of the expected cash flow, it was decided that an estimate of $20,000 was the most appropriate one. In other words, the judgment was that, letting* Y_0 *be the cash flow most actually reflects the estimator's subjective probabilities if σ is chosen as $20,000.*

$$Prob\,\{-\$400{,}000 - \sigma \leq Y_0 \leq -\$400{,}000 + \sigma\} = 0.6827$$

$$Prob\,\{-\$400{,}000 - 2\sigma \leq Y_0 \leq -\$400{,}000 + 2\sigma\} = 0.9545$$

$$Prob\,\{-\$400{,}000 - 3\sigma \leq Y_0 \leq -\$400{,}000 + 3\sigma\} = 0.9973$$

Proceeding with a similar analysis, the expected values and standard deviations of the net cash flows for each of the next five years were estimated. Due to the previous experience with model A, these standard deviations were considered to be small. The variation that does exist largely arises from the variation in the production costs, such as maintenance, equipment replacement, and rework costs, and in the state of the economy. Since these conditions tend to vary randomly from year to year, it was decided that the appropriate assumption is that the net cash flows in the various years are mutually independent. One special problem was encountered in determining the standard deviation for the fifth year since this net cash flow combines the regular cash flow for the fifth year plus the effective salvage value of the equipment being used. This standard deviation was obtained by assuming independence, so *that the variance of the sum equals the sum of the variances of these cash flows. Thus, even though the standard deviation for the salvage value was estimated at $30,000 and the standard deviation for the rest of the net cash flow at $40,000, the estimated standard deviation of the total net cash flow for the fifth year is $50,000. Table 2 summarizes the results of the estimating procedure for model A.*

Table 2 . Estimated Net Cash Flow Data in Thousands of Dollars for Model A by Hillier [22]

Year	Cash Flow Symbol	Expected Value	Standard Deviation
0	Y_0	-400	20
1	Y_1	120	10
2	Y_2	120	15
3	Y_3	120	20
4	Y_4	110	30
5	Y_5	200	50

The procedure for describing the investment in model B was similar. The primary difference was that this investment was considered to generate both a series of correlated cash flows and a series of independent cash flows. Thus, in equation 11, $m = 1$ instead of $m = 0$ as for the investment in model A. This difference arose because of the uncertainty regarding the reception of model B on the market. Thus, it was felt that if the reception exceeded expectations during the first year or two, it would continue to exceed present expectations thereafter, and vice versa. The resulting conclusion was that the net marketing cash flow, i.e., the net cash flow resulting from the sales income minus the expenses due to the marketing effort and advertising required, for each of the five years should be assumed to be perfectly correlated. On the other hand, it was felt the analysis of the production expenses involved was sufficiently reliable that any deviation from expectations for a given year would be primarily attributable to random fluctuations in production costs, especially in such *irregular items as maintenance costs. Therefore, it was concluded that the net production cash flow for each of the five years should be assumed to be mutually independent. The effective equipment salvage value, being essentially independent of the other cash flows, was included in the net production cash flow for the fifth year.*

Detailed analyses of the various components of total cash flow led, as for model A, to the desired estimates of the expectations and standard deviations of net cash flow for marketing and for production for each of the five years, as well as for the immediate investment required. These results are summarized in Table 3.

Table 3 . Estimated Net Cash Flow Data in Thousands of Dollars for Model B by Hillier [22]

Year	Source of Cash Flow	Cash Flow Symbol	Expected Value	Standard Deviation
0	Initial Investment	Y_0	-600	50
1	Production	Y_1	-250	20
2	Production	Y_2	-200	10
3	Production	Y_3	-200	10
4	Production	Y_4	-200	10
5	Production; Salvage Value	Y_5	-100	$10\sqrt{10}$
1	Marketing	$Z_1^{(1)}$	300	50
2	Marketing	$Z_2^{(1)}$	600	100
3	Marketing	$Z_3^{(1)}$	500	100
4	Marketing	$Z_4^{(1)}$	400	100
5	Marketing	$Z_5^{(1)}$	300	100

In the procedure to calculate the PDF of the IRR, i is assumed to be 10%.

For Model A, the variables are mutually independent. Therefore equations 9 and 10 are applied to find the mean and the variance of the IRR.

$$\mu_p = \sum_{j=0}^{5}\left[\frac{E(Y_j)}{(1+i)^j}\right] = -400 + \cdots + \frac{200}{1.1^5} = 95$$

$$\sigma_p^2 = \sum_{j=0}^{5}\left[\frac{Var(Y_j)}{(1+i)^{2j}}\right] = 20^2 + \cdots + \frac{50^2}{1.1^{10}} = 2247$$

When cumulative normal distribution table is used, the probability that

$$Prob(P < 0 | i = 10\%) = P\left(\frac{0-95}{\sqrt{2247}}\right) = 0.023$$

$$Prob(R < 10\%) = 0.023$$

Another numerical computation is performed here with the same data but i=%15 in order to evaluate how the probability changes with the changing rate of discount.

$$\mu_p = \sum_{j=0}^{5}\left[\frac{E(Y_j)}{(1+i)^j}\right] = -400 + \cdots + \frac{200}{1.15^5} = 36.3$$

$$\sigma_p^2 = \sum_{j=0}^{5}\left[\frac{Var(Y_j)}{(1+i)^{2j}}\right] = 20^2 + \cdots + \frac{50^2}{1.15^{10}} = 1689$$

When cumulative normal distribution table is used, the probability that

$$Prob(P < 0 | i = 10\%) = P\left(\frac{0-36.3}{\sqrt{1689}}\right) = 0.188$$

$$Prob(R < 10\%) = 0.188$$

For Model B, the mean and the variance are calculated from equation 12 and 13.

$$\mu_p = \sum_{j=0}^{5}\left[\frac{E(Y_j) + \sum_{k=1}^{m} E\left(Z_j^{(1)}\right)}{(1.1)^j}\right] = -600 + \frac{50}{1.1} + \cdots + \frac{200}{1.1^5} = 262$$

$$\sigma_p{}^2 = \sum_{j=0}^{n}\left[\frac{Var(Y_j)}{(1+i)^{2j}}\right] + \sum_{k=1}^{m}\left(\sum_{j=0}^{n}\left[\frac{\sqrt{Var\left(Z_j^{(k)}\right)}}{(1+i)^j}\right]\right)^2$$

$$= 2500 + \cdots + \frac{1000}{1.1^{10}} + \left(\frac{50}{1.1} + \cdots + \frac{100}{1.1^5}\right)^2 = 114{,}700$$

$$Prob(P < 0|i = 10\%) = P\left(\frac{0 - 262}{\sqrt{114700}}\right) = 0.22$$

$$Prob(R < 10\%) = 0.22$$

So if the same procedure is performed for various values of *i* from zero to infinity, it will be possible to obtain the CDF of the IRR. The graphs of the CDF and PDF of IRR are displayed in Figure 3 and Figure 4.

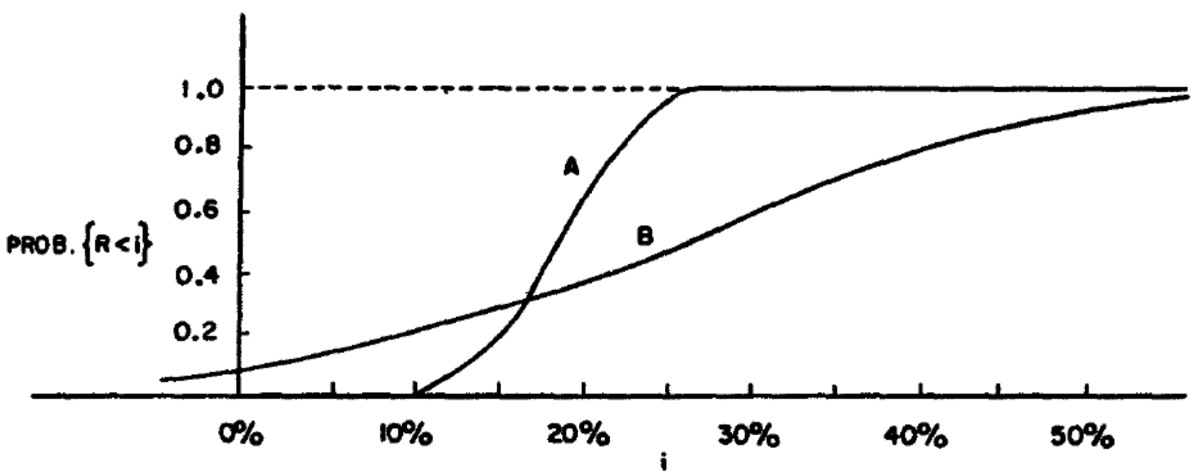

Figure 3 . Comparison of CDF of R for the Investments in Model A and in Model B by Hillier [22]

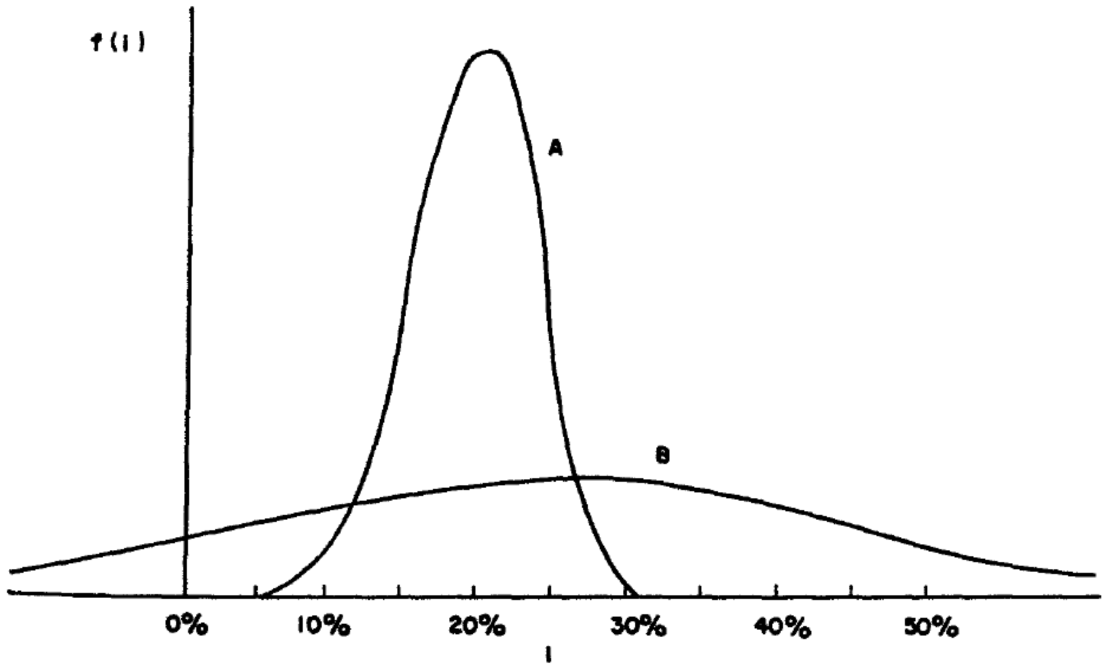

Figure 4 . Comparison of the Approximate PDF of R for the Investments in Model A and Model B by Hillier [22]

Figure 3 and Figure 4 depict rare information in stochastic engineering economics literature dominated by the PDF and the CDF of NPV.

Bernhard [2] commented on Hillier's article [22] considering the limiting assumptions. In Hillier's article, it is assumed that the highest IRR is always prioritized in the selection between the investments and that a higher expected value of IRR and the same or lower variance should be favored. However, Bernhard [2] illustrates that these recommendations may not hold in general with numerical examples. Also Bernhard states that if a reinvestment rate is used in the model, internal rate method is not correct in general as the IRR is completely independent with the reinvestment rate but the selection decision between the alternatives is not independent of that rate.

Keeley and Westerfield [28] states that Hiller's [22] method to find the probability of NPV misstates the value of the distant cash flows relative to the early ones which results with large errors in computation. Hodges and Moore [25] also points out that equation 14 may not be generally true but his method “is good enough for practical purposes.”

Hillier’s supplement [23] to his previous article [22] replies to the comments on and clarifies some critical issues. That equation 14 showing that NPV should be less than zero in order to obtain the cumulative density may not hold in general is the most important issue addressed to his paper. His comment to clarify this issue is to be concerned where the IRR is in the interval $[-1, \infty]$ as it will avoid the possibility that there are no valid values for R. Another comment is that a sufficient condition for equation 14 to hold is to have a joint probability function of all the cash flow such that NPV formula is a monotonic strictly decreasing function of *i* with probability one.

Rothkopf [40] suggests the usage of r' which would set the expected value of return $p(r') = 0$ instead of the $p(r) = 0$ as in Hillier’s article. This suggestion is suggested to avoid the misleading when there is correlation between the size of the project and its profitability. r' may not be present when r is not present as well. But this value will not be affected by the correlation between the project size and profitability.

However, Bernhard [2] also comments on Rothkopf’s [40] suggestion for this new index r'. Similar to the internal rate method, expected value of return method will have the same weakness as the independence from the reinvestment rate is not satisfied.

Another article about the PDF of the IRR is by Fairley and Jacoby [13]. The importance of the article is being one of the fewest articles that relate the investment analysis to the PDF of the IRR. In their article, the PDF of the IRR of an investment is calculated when cash flows are following a multivariate normal distribution. The exact PDF for the rate of return therefore approximated to normal distribution when variances of the cash flows are small. In addition to the normal probability model suggested, Fairley and Jacoby [13] stated an extended application of the analysis using a World Bank appraisal of a hydroelectric power investment in Argentina.

Let s_i be the values of the cash flows which are assumed to be coming from multivariate normal distribution. The means and the variances are also specified for each s_i and the covariances or the correlations between each pair of s_i. c_{ij}' s are the components which are random variables that are normal with the means, variances and correlations. These components should add up linearly as the resulting random variables s_i are normal random variables. r is the rate of return and the NPV is a function of r.

The NPV formula is changed into a n^{th} degree polynomial by assuming $x = 1/(1+r)$.

$$p(x) = \sum_{j=0}^{n} s_j x_j = s_0 + s_1 x + s_2 x^2 + \cdots + s_n x^n \tag{15}$$

A “mean polynomial” is constructed with the usage of the expected value of the s_i’s. The assumed expected values are denoted by $\overline{s_i}$.

$$m(x) = \sum_{j=0}^{n} \overline{s_j} x_j = \overline{s_0} + \overline{s_1} x + \overline{s_2} x^2 + \cdots + \overline{s_n} x^n \tag{16}$$

The problem is assumed to have a single root. The problem of interest is to associate the root r of p(x) with the root r_0 of m(x). This situation is maintained by g(r) which is called the "quasi-density" function that shows the density of the location of the root over the range of $-\infty < r < -1 , -1 < r < \infty$. In this article it is the density function for r.

A polynomial has a root in a small interval $(x + \Delta x)$ if and only if $\Delta p(x)/p(x) < -1$. The probability of being in this interval is $M(x, \Delta x) = P[\Delta p(x)/p(x)] < -1$.

f(x), which is the density for the location of a root is obtained as the limit:

$$f(x) = \lim_{\Delta x \to 0} (M(x, \Delta x)/\Delta x) = \left. \frac{dM(x, \Delta x)}{d(\Delta x)} \right|_{\Delta x = 0} \quad (17)$$

So when the derivation above and the transformation of variables are done for r, the probability density function turns out to be as in equation 18.

$$g(r) = \phi\left(\frac{m}{\sigma_u}\right)\left(\frac{\sigma_v}{\sigma_u}\right)(1 - c^2)^{\frac{1}{2}}\big(2\phi(\eta) + \eta(2\Phi(\eta) - 1)\big)\left(\frac{1}{(1 + \mathrm{r})^2}\right) \quad (18)$$

In the interval $-\infty < r < -1 , -1 < r < \infty$ and where

$$u = p\left(\frac{1}{1 + r}\right), v = p'\left(\frac{1}{1 + r}\right), \sigma_u = \sqrt{(var\ of\ u)}, \sigma_v = \sqrt{(var\ of\ v)},$$

$$c = (corr\ of\ u\ and\ v) = (cov\ of\ u\ and\ v)/(\sigma_u\sigma_v), \eta = (\frac{1}{1 - c^2})^{\frac{1}{2}}(\frac{\sigma_u m' - cm\sigma_v}{\sigma_u\sigma_v})$$

ϕ and Φ are the standard normal probability density and cumulative distribution functions respectively. p' and m' are the derivatives of p and m with respect to r.

As stated before, Fairley and Jacoby gave an analysis of a sample project in order to use the information given above. They used the data typical of the investment studies conducted by the World Bank. The project is defined as below directly taken from the reference [13].

The particular case is the El Chocon-Cerros Colorados hydroelectric scheme in Argentina. The analysis concerns the comparison of the hydroelectric development with alternative thermal power, on the assumption that growing market demand must be met one way or the other. Typically in analysis of this kind of capital intensive project the "cost" is the outlay on the hydroelectric scheme, and the "benefit" is the thermal system costs avoided if the hydroelectric project is built. The net difference between the two cost streams (thermal minus hydro) yields a series with negative numbers in the early years and positive numbers in the latter years. The internal rate of return on this cost stream is referred to as the "rate of return over cost" on the hydroelectric investment. Depending on the case, this rate may be compared with the rate of return in other sectors of the economy, or it may be matched against the cost of loan funds. If the rate of return is high enough, then the capital intensive hydroelectric scheme is recommended; if the return is too small, the project is rejected. In short, the rate of return is the decision variable for project choice. Similar calculations are used in the evaluation of capital intensive nuclear power plants.

Data contains the average cost components for the hydro and thermal plans and the net cost difference between the two plans from year 1 to 60. The cost components are the capital, fuel and the maintenance and operating costs. The corresponding differences between each cost components for each scheme are also specified. Three different random percentage errors for capital outflows in hydroelectric scheme, thermal scheme and for fuel costs are defined as $\sigma_d, \sigma_g, \sigma_e$ respectively. α is defined to be multiplier in the calculation of the fuel costs.

In this sample project the values are assigned as $\sigma_d = 0.15, \sigma_g = 0.01, \sigma_e = 0.10,$ and $\alpha = 1.024.$ The results are shown in Table 4 with the rate of return and the approximate CDF. Figure 5 shows the PDF of IRR.

The mean and standard deviation of g (r) are found by numerical integration of g (r) as:

$$E(r) = 0.0722 \quad \sigma_r = 0.0217$$

Table 4 . Density and CDF of Rate of Return Distribution by Fairley and Jacoby [13]

Rate of Return (r)	Density g(r)	CDF
0.015	0.50	0.005
0.030	1.59	0.021
0.045	5.05	0.072
0.060	12.77	0.214
0.075	19.41	0.478
0.090	15.66	0.751
0.105	7.40	0.912
0.120	2.53	0.974
0.135	0.75	0.994
0.150	0.22	1.000

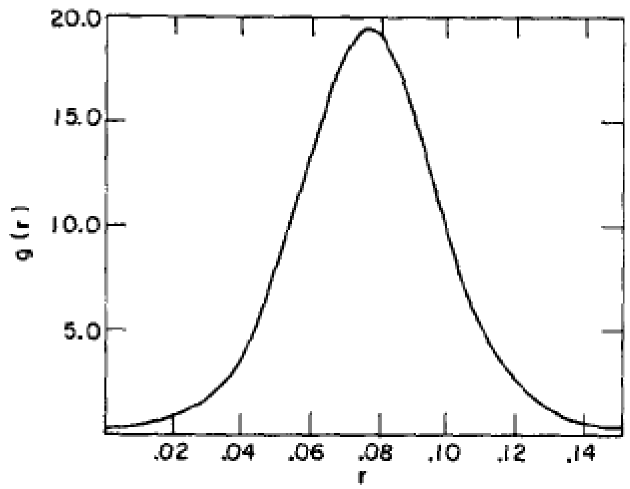

Figure 5 . Distribution of Rate of Return by Fairley and Jacoby [13]

Kaplan and Barish [27] have a different approach which treats the minimum acceptable internal rate of return as a random variable. In the examples illustrated in their article, Hillier's method [22] is used to calculate the PDF of IRR when the cash flows are normal random variables. However they suggest that, *i*, the minimum acceptable rate of return that is used to calculate the probability in equation 14 can be chosen a random variable R^* as well as a fixed value. This R^* is assumed to be independent of the R. When the results of the probabilities from both the constant and random minimum acceptable rate of returns are compared, they conclude that the random case differs significantly from the constant case due to the variance of the random variable. Whenever the minimum acceptable rate of return is relatively stable, its randomness does not affect the analysis very much.

Wagle [45] also introduced a similar method to Hillier's in order to derive the PDF of the IRR. However in his model, the true means and the variances of the cash flows did not have to be estimated by the user. Instead, the estimator is required to use an approach similar to the well known PERT technique. The optimistic (a), most likely (m) and pessimistic values (b) are estimated for each cash flow. The mean of the corresponding distribution is (a+b+4m)/6 and the standard deviation is (a-b)/6. In the example specified by the author, the mean and the standard deviation of the variables (e.g. market size, selling price, market growth rate, share of market, investment required, residual value, operating costs, fixed costs, useful life) are calculated with this method. If there are any, the correlations between these variables are calculated. The sales and the variable and fixed costs cash flows are found in terms of mean ad standard deviation. The net cash flows are found when the investment costs, sales cash flow, cost cash flows and residual values are combined for each period. The NPV is calculated and the PDF of IRR is determined with equation 14 which is also the method used by Hiller [22].

Some authors study calculation of the IRR. Pohjola and Turunen [34] try to estimate the internal rate of return from the fuzzy data. Ranasinghe and Russell [36] suggest an analytical approach to Monte Carlo simulation for finding the internal rate of return.

Besides the analytical methods to find the PDF of IRR, some authors, such as Hertz [19], Elnicki [11], and Lewellen [29] solved the same problem with simulation.

Hertz [19] developed a Monte Carlo simulation method to find the PDF of IRR when the cash flows are functions of several variables each having their own probability distributions. In his analysis, range of values and the expectation of each variable are estimated. In the simulation, each variable is assigned a random value according to their distributions and these values are combined to compute the internal rate of return. The same procedure is repeated in order to find a clear portrayal of the investment risk.

Bower and Lessard [3] discuss different risk screening processes in the capital budgeting problems and show how these methods may mismatch the market criteria and the project measures. They also suggest a new effective method that finds the systematic risk of a project with a ratio that includes the expected value and the standard deviation of IRR, and a correlation term with the market criteria.

Moore and Chen [31] oppose Hertz's [19] and Hillier's [22] classical idea that the true parameters of cash flows can be estimated. They suggest a simulation method based on the distributions that are predicted from sample and prior information about the parameters. They also compare the simulation results with the classical model. Sarper [41] uses the Newton's method in a simulation model to find the parameters of the PDF of IRR. The equations and examples to derive the IRR of simple and non simple investments with Newton's method are provided. An example to incorporate uncertain project duration, cash flows and tax rates into the simulation code is also presented.

Robichek [38] discusses two different errors in the areas of risk analysis. First error is about the Hillier's [22] conclusion that the PDFs of IRR and NPV are exactly the same except for the mean and the standard deviation. The second error is the interpretation that when the PDF of IRR is obtained by simulation, the expected rate of return of the problem and the mean of the distribution of the simulated IRR's are identical. With the examples illustrated, he shows that there may be differences between the forms of distributions of NPV and IRR and the expected IRR is not generally equal to the mean of the distribution of the simulated IRRs.

Carmichael and Balatbat [5] present a survey on the field of the probabilistic cash flow models including the methods of IRR, NPV, annual worth, future worth, payback period and benefit cost ratios . According to their survey, the recent research on the PDF of IRR is given by Kahraman [26] who presents a numerical example in this article as shown below:

If the net cash flows are as in the Table 5 below.

Table 5 . Net Cash Flow Parameters by Kahraman [26]

Year	**$E[A_j]$**	**$V[A_j]$**
0	-I	V(I)
1	+ A_1	$V(A_1)$
2	$+A_2$	$V(A_2)$
3	$+A_3$	$V(A_3)$
…	…	…
N	$+A_N$	$V(A_N)$

The PDF of the IRR would be given by equation 19.

$$\sum_{j=1}^{N} E\left[A_j\right](1 - i_0)^{-j} - I \leq 0 \tag{19}$$

In their example, the cost of capital is assumed to be 10%.

Table 6 . Example Net Cash Flow Table by Kahraman [26]

Year	$E[A_j]$	$V[A_j]$
0	-$10000	$1*10^6$
1-10	$1800	$4*10^4$

$$\Pr(PV \leq 0) = \Pr\left(z \leq \frac{0 - \left(-\$10000 + \sum_{j=1}^{10} \$1800(1.10)^{-j}\right)}{1 * 10^6 + (4 * 10^4) \sum_{j=1}^{10} (1.10)^{-2j}}\right) =$$

$$\Pr\left(z \leq \frac{0 - 1059.20}{1078.04}\right) = 0.163$$

Using equation 19,

$$\sum_{j=1}^{10} \$1800\,(1 - i^*)^{-j} - 10000 = 0$$

$$\$1800(P/A, i^*, 10\%) - 10000 = 0$$

Where i^* is found to be 12.5%. So for $i \geq 12.5$, the net present value is equal to zero or smaller than 0. This means that the capital cost of the firm has a probability of 16.3% to increase from 10% to 12.5% or more.

This thesis deals with finding the probability distribution of functions of random variables. Therefore it includes the quotient, product and sum of the random variables.

Curtiss' [7] article gives detailed computations to find the distribution of the quotient of two chance variables. He also proves when the quotient distribution exists and assumes that the two chance variables are independent from each other. Geary [16] does a similar study by finding the distribution of the quotient of two normal variables. Marsaglia [30] also finds the PDF of the ratios of normal variables. He also calculates the ratios of the sums of independent uniform variables. Fieller [15] finds the distribution of the index in a normal bivariate population. Hayya, Armstrong, and Gressis [18] show that a suitable transformation of the ratio of two normally distributed variables is approximately normally distributed when the coefficient of variation of the denominator is less than 0.39. Hinkley [24] discusses the distribution of the ratio of two correlated random variables. The exact distribution and the approximation are also compared.

Derivation of the PDF of IRR for two period problems is mainly related with the distribution of the roots of polynomials. Erdos and Turan [12] and Hamblen [20]provided studies for this aspect of the PDF of IRR. Fieller [14] points out some problems in the determination of the distributions of the roots of simple equations and quadratic equations.

Hamblen [20] found the distributions of roots of quadratic equations with random coefficients. The roots may be real or complex. The roots of interest for this thesis are the real roots. Distribution function of the real roots of a quadratic function are found analytically and two examples where the coefficients are bivariate normal and gamma random variables.

For the quadratic function, the equation is written in the form of $\eta^2 - \varepsilon_1\eta + \varepsilon_2 = 0$ where ε_1 and ε_2 are the random variables. The roots η_1 and η_2 are found with following relationships.

$$\eta_1 = \frac{\varepsilon_1}{2} + \sqrt{\frac{\varepsilon_1^2}{4} - \varepsilon_2} \qquad \eta_2 = \frac{\varepsilon_1}{2} - \sqrt{\frac{\varepsilon_1^2}{4} - \varepsilon_2} \qquad (20)$$

The real roots of the quadratic functions are obtained in the interval $\varepsilon_2 \leq \varepsilon_1^2/4$. The probability of the real roots and the conditional PDFs are given by equations 21 and 22 respectively.

$$P(R) = P\left(\varepsilon_2 \leq \frac{\varepsilon_1^2}{4}\right) = \iint_{y \leq \frac{x^2}{4}} f(x,y)dydx \qquad (21)$$

$$f(x,y|R) = \frac{f(x,y)}{P(R)}, \qquad y \leq \frac{x^2}{4} \qquad (22)$$

When the change of variables is used, $g(v_1, v_2|R)$, the PDF of the real roots, is given with equation 23.

$$g(v_1, v_2|R) = \frac{f(v_1 + v_2, v_1 v_2)|J|}{P(R)for} \text{ for all } v_1 \geq v_2 \text{ and } |J| = (v_1 - v_2) \qquad (23)$$

The marginal density functions of real roots η_1 and η_2 are given by $g(v_1|\mathrm{R})$and $g(v_2|\mathrm{R})$:

$$g_1(v_1|\mathrm{R}) = \int_{-\infty}^{v_1} g(v_1, v_2|\mathrm{R})\ dv_2 \qquad (24)$$

$$g_2(v_2|\mathrm{R}) = \int_{v_1}^{\infty} g(v_1,\ v_2|\mathrm{R})\ dv_1 \qquad (25)$$

As stated previously, Hamblen's [20] results are relevant to this thesis but no direct connection could be made.

Chapter 3 – Methodology

This thesis seeks to find a non-simulation and a non-iterative solution for the PDF of IRR in two specific problems, one and two period cases. Firstly, the PDF of IRR is derived when only one cash flow is a random variable. All possible combinations are tried by selection of one random cash flow at a time. In the next step, the aim is to derive the PDF of IRR when there is more than one random cash flow in the problem. Also the problem is calculated for each combination twice assuming the random variables follow exponential or uniform distributions at a time. The problems are solved with both transformation of random variables and distribution function methods if possible. Detailed information on the distribution function and transformation of random variables methods can be found in the references [1], [6], [9], [17], [33], [35], [39], [43].

3.1 One Period IRR Problem: A_0 Constant, A_1 Uniform

$$i = A_1/A_0 - 1$$

In equation 2, A_0 is constant and assumed to be positive by default. For the initial cash flow, - A_0 is used. A_1 is uniformly distributed between $[a_1, b_1]$ with the PDF of $f(A_1) = 1/(b_1 - a_1)$.

3.1.1. Derivation of the PDF with the Distribution Function Method

$$P\left(\frac{A_1}{A_0} - 1 \leq i\right) = P(A_1 \leq (i+1)A_0) = F_{A_1}\left((i+1)A_0\right) = \frac{(i+1)A_0 - a_1}{b_1 - a_1}$$

$$F(i) = \frac{(i+1)A_0 - a_1}{b_1 - a_1} \tag{26}$$

where the limits are:

$$\frac{a_1}{A_0} - 1 \le i \le \frac{b_1}{A_0} - 1$$

$$f(i) = \frac{(i+1)A_0 - a_1}{b_1 - a_1} di = \frac{A_0}{b_1 - a_1} \tag{27}$$

3.1.2. Derivation of the PDF with the Transformation Method

Transformation for cx=u has a resulting density function

$$h(u) = \left|\frac{1}{c}\right| f\left(\frac{u}{c}\right) \tag{28}$$

Let

$$\frac{A_1}{A_0} = u \qquad \frac{1}{A_0} = c \qquad \frac{u}{c} = A_1$$

$$h(u) = \left|\frac{1}{\frac{1}{A_0}}\right| f(A_1) = A_0 * \frac{1}{b_1 - a_1} \tag{29}$$

where the limits are $\left[\frac{a_1}{A_0}, \frac{b_1}{A_0}\right]$

Let $x = \frac{A_1}{A_0}$ and $c = -1$.

$$\mathrm{f(u-c)} = \mathrm{f(x)} = f\left(\frac{A_1}{A_0}\right) = \boldsymbol{A_0} * \frac{\mathbf{1}}{\boldsymbol{b_1 - a_1}} \tag{30}$$

where the limits change to: $\left[\frac{a_1}{A_0} - 1, \frac{b_1}{A_0} - 1\right]$

The CDF is checked for validity.

$$\int_{\frac{a_1}{A_0}-1}^{\frac{b_1}{A_0}-1} A_0 \frac{1}{b_1 - a_1} du = A_0 \frac{1}{b_1 - a_1}\left[\frac{b_1 - A_0}{A_0} - \frac{a_1 - A_0}{A_0}\right] = A_0 \frac{1}{b_1 - a_1} * \frac{b_1 - a_1}{A_0} = 1$$

3.1.3. The Expected Value

$$\int_{\frac{a_1}{A_0}-1}^{\frac{b_1}{A_0}-1} A_0 \frac{1}{b_1 - a_1} idi = \frac{A_0}{2}\frac{1}{b_1 - a_1}\left[(\frac{b_1 - A_0}{A_0})^2 - (\frac{a_1 - A_0}{A_0})^2\right]$$

$$= \frac{A_0}{2}\frac{1}{b_1 - a_1} * \frac{b_1 - a_1}{A_0} * \frac{b_1 + a_1 - 2A_0}{A_0} = \frac{b_1 + a_1 - 2A_0}{2A_0}$$

$$E(i) = \frac{\boldsymbol{b_1 + a_1}}{\boldsymbol{2A_0}} - \mathbf{1} \tag{31}$$

3.1.4. The Variance

$$\int_{\frac{a_1}{A_0}-1}^{\frac{b_1}{A_0}-1} A_0 \frac{1}{b_1 - a_1} i^2 di = \frac{A_0}{3}\frac{1}{b_1 - a_1}\left[(\frac{b_1 - A_0}{A_0})^3 - (\frac{a_1 - A_0}{A_0})^3\right] - (\frac{b_1 + a_1}{2A_0} - 1)^2 =$$

$$= \frac{A_0}{3}\frac{1}{b_1 - a_1}\left[(\frac{b_1 - a_1}{A_0})^3 + 3(\frac{b_1 - A_0}{A_0})(\frac{a_1 - A_0}{A_0})(\frac{b_1 - a_1}{A_0})\right]$$

$$- \left[(\frac{b_1 + a_1}{2A_0})^2 - 2\frac{b_1 + a_1}{2A_0} + 1\right]$$

$$= \frac{A_0}{3}\frac{1}{b_1 - a_1} * \frac{b_1 - a_1}{{A_0}^3} * [(b_1 - a_1)^2 + 3(b_1 - A_0)(a_1 - A_0)]$$

$$- \left[\frac{(b_1 + a_1)^2}{4{A_0}^2} - \frac{b_1 + a_1}{A_0} + 1\right]$$

$$= \frac{1}{3{A_0}^2} * [(b_1 - a_1)^2 + 3(b_1 - A_0)(a_1 - A_0)] - \left[\frac{(b_1 + a_1)^2}{4{A_0}^2} - \frac{b_1 + a_1}{A_0} + 1\right]$$

$$= \frac{(b_1 - a_1)^2}{3{A_0}^2} - \frac{(b_1 + a_1)^2}{4{A_0}^2} + \frac{3(b_1a_1 - A_0a_1 - A_0b_1 + {A_0}^2)}{3{A_0}^2} + \frac{b_1 + a_1}{A_0} - 1$$

$$= \frac{4(b_1 - a_1)^2 - 3(b_1 + a_1)^2}{12{A_0}^2} + \frac{b_1a_1}{{A_0}^2} - \frac{(a_1 + b_1)}{A_0} + 1 + \frac{b_1 + a_1}{A_0} - 1$$

$$= \frac{4{b_1}^2 - 8b_1a_1 + 4{a_1}^2 - 3{b_1}^2 - 6b_1a_1 - 3{a_1}^2}{12{A_0}^2} + \frac{b_1a_1}{{A_0}^2}$$

$$= \frac{{b_1}^2 - 14b_1a_1 + {a_1}^2}{12{A_0}^2} + \frac{b_1a_1}{{A_0}^2} = \frac{{b_1}^2 + {a_1}^2}{12{A_0}^2} - \frac{b_1a_1}{6{A_0}^2}$$

$$\frac{1}{6{A_0}^2}\frac{{b_1}^2 + {a_1}^2 - 2b_1a_1}{2}$$

$$\sigma^2 = \frac{\mathbf{1}}{\mathbf{12}{\boldsymbol{A_0}}^{\mathbf{2}}}(\boldsymbol{b_1} - \boldsymbol{a_1})^{\mathbf{2}} \qquad (32)$$

3.2 One Period IRR Problem: Both A_0 and A_1 Uniform

A_0 is uniformly distributed between $[a_0, b_0]$ and assumed to be positive by default. For the initial cash flow, - A_0 is used. A_1 is uniformly distributed between $[a_1, b_1]$.

$$f(A_0) = \frac{1}{b_0 - a_0} \qquad f(A_1) = \frac{1}{b_1 - a_1}$$

In this problem, the ranges $[a_0, b_0]$ and $[a_1, b_1]$ are selected so that they are not overlapping. In case these ranges are overlapping, the intervals where the IRR can exist as found in the next section change significantly.

3.2.1. Derivation of the PDF with the Distribution Function Method

The distribution of $\frac{A_1}{A_0}$ is found first. Figure 6 shows how the intervals are determined for the quotient of uniform random variables.

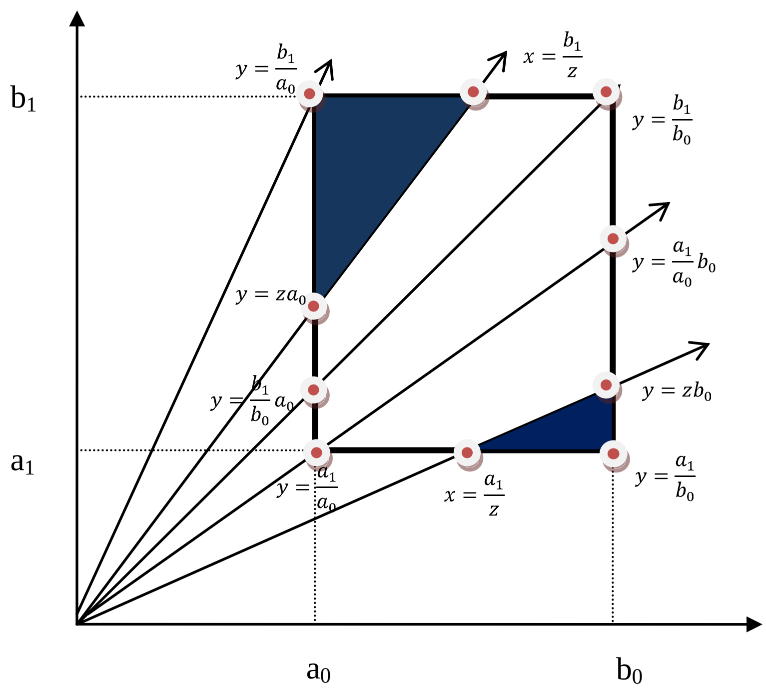

Figure 6 . Graphical Representation of Possible Regions for One Period Problem with All Random Cash Flows

As the uniform distributions have upper and lower bounds, the intervals that quotient of the random variables exist should be determined.

Interval 1: $\boldsymbol{\frac{a_1}{b_0} < z < \frac{a_1}{a_0}}$

$$P\left(\frac{y}{x} \leq z\right) = \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} area$$

$$base = b_0 - \frac{a_1}{z} \qquad height = zb_0 - a_1$$

$$area = \frac{1}{2}(b_0 - \frac{a_1}{z})(zb_0 - a_1)$$

$$F(z) = \frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\frac{1}{2}(b_0 - \frac{a_1}{z})(zb_0 - a_1)$$

$$= \frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\frac{1}{2}\left({b_0}^2 z - b_0 a_1 - b_0 a_1 + \frac{{a_1}^2}{z}\right)$$

$$= \frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\frac{1}{2}\left({b_0}^2 z - 2b_0 a_1 + \frac{{a_1}^2}{z}\right)$$

$$f(z) = \frac{1}{b_0 - a_0}\ \frac{1}{b_1 - a_1}\ \frac{1}{2}\left({b_0}^2 - \frac{{a_1}^2}{z^2}\right) \quad (33)$$

Interval 2: $\boldsymbol{\frac{a_1}{a_0}} < z < \boldsymbol{\frac{b_1}{b_0}}$

$$P\left(\frac{y}{x} \le z\right) = \frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\frac{1}{2}\left[\left(zb_0 - \frac{a_1 b_0}{a_0} + za_0 - \frac{a_1 a_0}{a_0}\right) + \left(\frac{a_1 b_0}{a_0} - a_1\right)\right](b_0 - a_0)$$

$$F(z) = \frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\frac{1}{2}\left[\left((b_0 + a_0)(z - \frac{a_1}{a_0}\right) + \frac{a_1}{a_0}(b_0 - a_0)\right](b_0 - a_0)$$

$$= \frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\frac{1}{2}\left[\left((b_0 + a_0)(z - \frac{a_1}{a_0}\right) + \frac{a_1}{a_0}(b_0 - a_0)\right](b_0 - a_0)$$

$$= \frac{1}{b_1 - a_1}\frac{1}{2}\left[\left((b_0 + a_0)(z - \frac{a_1}{a_0}\right) + \frac{a_1}{a_0}(b_0 - a_0)\right]$$

$$= \frac{1}{b_1 - a_1}\frac{1}{2}\left[(b_0 + a_0)z - \frac{a_1}{a_0}(b_0 + a_0) + \frac{a_1}{a_0}(b_0 - a_0)\right]$$

$$F(z) = \frac{1}{b_1 - a_1}\frac{1}{2}[(b_0 + a_0)z - 2a_1]$$

$$f(z) = \frac{1}{2}\left(\frac{b_0 + a_0}{b_1 - a_1}\right) \quad (34)$$

Interval 3: $\frac{\boldsymbol{b_1}}{\boldsymbol{b_0}} < z < \frac{\boldsymbol{b_1}}{\boldsymbol{a_0}}$

$$P\left(\frac{y}{x} \le z\right) = 1 - \left[\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\frac{1}{2}\left(\frac{b_1}{z} - a_0\right)(b_1 - za_0)\right]$$

$$F(z) = 1 - \frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\frac{1}{2}\left(\frac{{b_1}^2}{z} - b_1a_0 - b_1a_0 + {a_0}^2 z\right)$$

$$F(z) = 1 - \frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\frac{1}{2}\left(\frac{{b_1}^2}{z} - 2b_1a_0 + {a_0}^2 z\right)$$

$$f(z) = -\frac{1}{2}\,\frac{1}{b_0 - a_0}\,\frac{1}{b_1 - a_1}\left({a_0}^2 - \frac{{b_1}^2}{z^2}\right) \tag{35}$$

Then these distributions are offset with the term -1 in equation 2. Table 7 summarizes the PDF expressions for each interval.

Table 7 . Summary of the PDF Expressions for One Period Problem with All Uniform Cash Flows

$f(i)$ or PDF	Interval
$\frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left({b_0}^2 - \frac{{a_1}^2}{(i+1)^2}\right)$	$\frac{a_1}{b_0} - 1 \le i < \frac{a_1}{a_0} - 1$
$\frac{1}{2}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)$	$\frac{a_1}{a_0} - 1 \le i < \frac{b_1}{b_0} - 1$
$-\frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left({a_0}^2 - \frac{{b_1}^2}{(i+1)^2}\right)$	$\frac{b_1}{b_0} - 1 \le i \le \frac{b_1}{a_0} - 1$

Table 8 shows the CDF expressions for each interval.

Table 8 . Summary of the CDF Expressions for One Period Problem with All Uniform Cash Flows

$F(i)$ or CDF	Interval
$\frac{1}{2}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}\left(b_0{}^2(i+1)-2b_0a_1+\frac{a_1{}^2}{(i+1)}\right)$	$\frac{a_1}{b_0}-1\leq i<\frac{a_1}{a_0}-1$
$\frac{1}{2}\frac{1}{b_1-a_1}[(b_0+a_0)(i+1)-2a_1]$	$\frac{a_1}{a_0}-1\leq i<\frac{b_1}{b_0}-1$
$1-\frac{1}{2}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}\left(\frac{b_1{}^2}{(i+1)}-2b_1a_0+a_0{}^2(i+1)\right)$	$\frac{b_1}{b_0}-1\leq i<\frac{b_1}{a_0}-1$

The CDF is checked for validity:

$$=\frac{1}{2}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}\left(b_0{}^2(i+1)-2b_0a_1+\frac{a_1{}^2}{(i+1)}\right)\Bigg|_{\frac{a_1}{b_0}-1}^{\frac{a_1}{a_0}-1}$$

$$+\frac{1}{2}\frac{1}{b_1-a_1}[(b_0+a_0)(i+1)-2a_1]\Bigg|_{\frac{a_1}{a_0}-1}^{\frac{b_1}{b_0}-1}$$

$$+\left[1-\frac{1}{2}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}\left(\frac{b_1{}^2}{(i+1)}-2b_1a_0+a_0{}^2(i+1)\right)\right]\Bigg|_{\frac{b_1}{b_0}-1}^{\frac{b_1}{a_0}-1}$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(b_0{}^2\left(\frac{a_1}{a_0} - 1 + 1\right) - 2b_0a_1 + \frac{a_1{}^2}{\left(\frac{a_1}{a_0} - 1 + 1\right)} - b_0{}^2\left(\frac{a_1}{b_0} - 1 + 1\right) + 2b_0a_1\right.$$

$$\left. - \frac{a_1{}^2}{\left(\frac{a_1}{b_0} - 1 + 1\right)}\right) + \frac{1}{2}\frac{1}{b_1 - a_1}\left((b_0 + a_0)\left(\frac{b_1}{b_0} - 1 + 1\right) - 2a_1 - (b_0 + a_0)\left(\frac{a_1}{a_0} - 1 + 1\right) + 2a_1\right)$$

$$+1 - \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{b_1{}^2}{\left(\frac{b_1}{a_0} - 1 + 1\right)} - 2b_1a_0 + a_0{}^2\left(\frac{b_1}{a_0} - 1 + 1\right)\right)$$

$$-1 + \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{b_1{}^2}{\left(\frac{b_1}{b_0} - 1 + 1\right)} - 2b_1a_0 + a_0{}^2\left(\frac{b_1}{b_0} - 1 + 1\right)\right)$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(b_0{}^2\left(\frac{a_1}{a_0}\right) + a_0a_1 - b_0a_1 - b_0a_1\right) + \frac{1}{2}\frac{1}{b_1 - a_1}\left((b_0 + a_0)\left(\frac{b_1}{b_0}\right) - (b_0 + a_0)\left(\frac{a_1}{a_0}\right)\right)$$

$$- \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}(b_1a_0 - 2b_1a_0 + b_1a_0) + \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(b_1b_0 - 2b_1a_0 + a_0{}^2\left(\frac{b_1}{b_0}\right)\right)$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(b_0{}^2\left(\frac{a_1}{a_0}\right) + a_0a_1 - b_0a_1 - b_0a_1 - b_1a_0 + 2b_1a_0 - b_1a_0 + b_1b_0 - 2b_1a_0\right.$$

$$\left. + a_0{}^2\left(\frac{b_1}{b_0}\right)\right) + \frac{1}{2}\frac{1}{b_1 - a_1}\frac{1}{b_0 - a_0}\left((b_0^2 - a_0^2)\left(\frac{b_1}{b_0} - \frac{a_1}{a_0}\right)\right)$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(b_0{}^2\left(\frac{a_1}{a_0}\right) + a_0a_1 - b_0a_1 - b_0a_1 - b_1a_0 + 2b_1a_0 - b_1a_0 + b_1b_0 - 2b_1a_0\right.$$

$$\left. + a_0{}^2\left(\frac{b_1}{b_0}\right) + b_0^2\frac{b_1}{b_0} - b_0^2\frac{a_1}{a_0} - a_0^2\frac{b_1}{b_0} + a_0^2\frac{a_1}{a_0}\right)$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left({b_0}^2 \left(\frac{a_1}{a_0} \right) + a_0 a_1 - b_0 a_1 - b_0 a_1 - b_1 a_0 + 2 b_1 a_0 - b_1 a_0 + b_1 b_0 - 2 b_1 a_0 \right.$$

$$\left. + {a_0}^2 \left(\frac{b_1}{b_0} \right) + b_0 b_1 - b_0^2 \frac{a_1}{a_0} - a_0^2 \frac{b_1}{b_0} + a_0 a_1 \right)$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} (a_0 a_1 - b_0 a_1 - b_0 a_1 - b_1 a_0 + 2 b_1 a_0 - b_1 a_0 + b_1 b_0 - 2 b_1 a_0 + b_0 b_1 + a_0 a_1)$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} (2 a_0 a_1 - 2 b_0 a_1 + 2 b_1 b_0 - 2 b_1 a_0)$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} 2 (b_0 - a_0)(b_1 - a_1) = 1$$

3.2.2. Derivation of the PDF with the Transformation Method

Transformation for the term A_1/A_0 is $h(u) = \int_{a_1}^{b_1} f(v) g(\frac{v}{u}) \left| \frac{v}{u^2} \right| dv$.

Assuming that $u = \frac{A_1}{A_0}$ and $v = A_1$, possible regions for $[a_1, b_1] \cap [a_0 u, b_0 u]$ are shown as in the Figure 7 below:

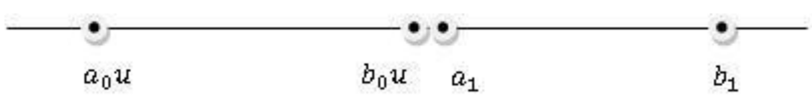

Figure 7 . Possible Regions for One Period Problem with All Random Cash Flows

Possible region 1: $[a_1, b_0 u]$ - - > $\frac{a_1}{b_0} < u < \frac{a_1}{a_0}$

Possible region 2: $[a_0 u, b_0 u]$ - - > $\frac{a_1}{a_0} < u < \frac{b_1}{b_0}$

Possible region 3: $[a_0 u, b_1]$ - - > $\frac{b_1}{b_0} < u < \frac{b_1}{a_0}$

$$f_1(u) = \int_{a_1}^{b_0 u} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \frac{v}{u^2} dv = \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \frac{v^2}{u^2} \bigg|_{a_1}^{b_0 u}$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \frac{{b_0}^2 u^2 - {a_1}^2}{u^2} = \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left[{b_0}^2 - \frac{{a_1}^2}{u^2} \right]$$

$$f_1(u-1) = \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left[{b_0}^2 - \frac{{a_1}^2}{(u+1)^2} \right]$$

$$f_1(i) = \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left[{b_0}^2 - \frac{{a_1}^2}{(i+1)^2} \right]$$

$$f_2(u) = \int_{a_0 u}^{b_0 u} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \frac{v}{u^2} dv = \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \frac{v^2}{u^2} \bigg|_{a_0 u}^{b_0 u}$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \frac{{b_0}^2 u^2 - {a_0}^2 u^2}{u^2} = \frac{1}{2} \frac{b_0 + a_0}{b_1 - a_1}$$

$$f_2(u-1) = f_2(i) = \frac{1}{2} \frac{b_0 + a_0}{b_1 - a_1}$$

$$f_3(u) = \int_{a_0 u}^{b_1} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \frac{v}{u^2} dv = \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \frac{v^2}{u^2} \bigg|_{a_0 u}^{b_1}$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \frac{{b_1}^2 - {a_0}^2 u^2}{u^2} = \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left[\frac{{b_1}^2}{u^2} - {a_0}^2 \right]$$

$$f_3(u-1) = -\frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left[{a_0}^2 - \frac{{b_1}^2}{(u+1)^2} \right]$$

$$f_3(i) = -\frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left[{a_0}^2 - \frac{{b_1}^2}{(i+1)^2} \right]$$

The results with transformation of random variables method is the same as the ones found by the distribution function method. For the summary of results, the reader is referred to Table 7 above.

3.2.3. The Expected Value

The following indefinite integral is needed in the course of the calculus work.

$$\int \frac{i}{(1+i)^2} di = \frac{1}{1+i} + \ln(1+i)$$

Let's integrate all three regions separately.

For $\frac{a_1}{b_0} - 1 < i < \frac{a_1}{a_0} - 1$ region,

$$E_1(i) = \int_{\frac{a_1}{b_0}-1}^{\frac{a_1}{a_0}-1} \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left({b_0}^2 - \frac{{a_1}^2}{(i+1)^2} \right) i \, di$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left[{b_0}^2 \frac{i^2}{2} - {a_1}^2 \left(\frac{1}{1+i} + \ln(1+i) \right) \right]_{\frac{a_1}{b_0}-1}^{\frac{a_1}{a_0}-1}$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left[{b_0}^2 \frac{\left(\frac{a_1}{a_0} - 1\right)^2}{2} - {a_1}^2 \left(\frac{1}{1 + \frac{a_1}{a_0} - 1} + \ln\left(1 + \frac{a_1}{a_0} - 1\right) \right) \right.$$

$$\left. - {b_0}^2 \frac{\left(\frac{a_1}{b_0} - 1\right)^2}{2} + {a_1}^2 \left(\frac{1}{1 + \frac{a_1}{b_0} - 1} + \ln\left(1 + \frac{a_1}{b_0} - 1\right) \right) \right]$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left[{b_0}^2 \frac{\left(\frac{a_1}{a_0} - 1\right)^2}{2} - {b_0}^2 \frac{\left(\frac{a_1}{b_0} - 1\right)^2}{2} + {a_1}^2 \left(\frac{b_0}{a_1} + \ln\left(\frac{a_1}{b_0}\right) - \frac{a_0}{a_1} - \ln\left(\frac{a_1}{a_0}\right) \right) \right]$$

$$= \frac{1}{2} \frac{1}{b_0 - a_0} \frac{1}{b_1 - a_1} \left(\frac{{b_0}^2}{2} \left[\frac{(a_1 - a_0)^2 {b_0}^2 - (a_1 - b_0)^2 {a_0}^2}{{a_0}^2 {b_0}^2} \right] + {a_1}^2 \left[\frac{b_0 - a_0}{a_1} + ln\left(\frac{\frac{a_1}{b_0}}{\frac{a_1}{a_0}} \right) \right] \right)$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{1}{2a_0{}^2}\left[(a_1{}^2 - 2a_1a_0 + a_0{}^2)b_0{}^2 - \left(a_1{}^2 - 2a_1b_0 + b_0{}^2\right)a_0{}^2\right]\right.$$

$$\left. + a_1(b_0 - a_0) + a_1{}^2 ln\left(\frac{a_0}{b_0}\right)\right)$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{1}{2a_0{}^2}\left[\left(a_1{}^2b_0{}^2 - 2a_1a_0b_0{}^2 + a_0{}^2b_0{}^2\right)\right.\right.$$

$$\left.\left. - \left(a_1{}^2a_0{}^2 - 2a_1b_0a_0{}^2 + b_0{}^2a_0{}^2\right)\right] + a_1(b_0 - a_0) + a_1{}^2 ln\left(\frac{a_0}{b_0}\right)\right)$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{1}{2a_0{}^2}\left[a_1{}^2\left(b_0{}^2 - a_0{}^2\right) - 2a_1b_0a_0(b_0 - a_0)\right]\right. \quad (36)$$

$$\left. + a_1(b_0 - a_0) + a_1{}^2 ln\left(\frac{a_0}{b_0}\right)\right)$$

For $\frac{a_1}{a_0} - 1 < i < \frac{b_1}{b_0} - 1$ region,

$$E_2(i) = \int_{\frac{a_1}{a_0}-1}^{\frac{b_1}{b_0}-1} \frac{1}{2}\left(\frac{b_0 + a_0}{b_1 - a_1}\right) i\ di = \frac{1}{2}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left[\frac{i^2}{2}\right]_{\frac{a_1}{a_0}-1}^{\frac{b_1}{b_0}-1}$$

$$= \frac{1}{2}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left[\frac{\left(\frac{b_1}{b_0} - 1\right)^2}{2} - \frac{\left(\frac{a_1}{a_0} - 1\right)^2}{2}\right]$$

$$= \frac{1}{4}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left(\frac{b_1}{b_0} - 1 + \frac{a_1}{a_0} - 1\right)\left(\frac{b_1}{b_0} - 1 - \frac{a_1}{a_0} + 1\right)$$

$$= \frac{1}{4}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left(\frac{b_1}{b_0} + \frac{a_1}{a_0} - 2\right)\left(\frac{b_1}{b_0} - \frac{a_1}{a_0}\right)$$

$$= \frac{1}{4}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left(\frac{a_0 b_1}{a_0 b_0} + \frac{a_1 b_0}{a_0 b_0} - \frac{2a_0 b_0}{a_0 b_0}\right)\left(\frac{a_0 b_1}{a_0 b_0} - \frac{a_1 b_0}{a_0 b_0}\right)$$

$$= \frac{1}{4}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left(\frac{1}{{a_0}^2 {b_0}^2}\right)(a_0 b_1 + a_1 b_0 - 2a_0 b_0)(a_0 b_1 - a_1 b_0)$$

$$= \frac{1}{4}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left(\frac{1}{{a_0}^2 {b_0}^2}\right)({a_0}^2 {b_1}^2 + a_1 b_1 a_0 b_0 - 2{a_0}^2 b_0 b_1 - a_1 b_1 a_0 b_0 - {a_1}^2 {b_0}^2 + 2{b_0}^2 a_0 b_1)$$

$$= \frac{1}{4}\left(\frac{1}{b_1 - a_1}\right)\left(\frac{1}{{a_0}^2 {b_0}^2}\right)({a_0}^2 {b_1}^2 - {a_1}^2 {b_0}^2 + 2a_0 b_0(a_1 b_0 - a_0 b_1))(b_0 + a_0)$$

$$= \frac{1}{4}\left(\frac{1}{b_1 - a_1}\right)\left(\frac{1}{{a_0}^2 {b_0}^2}\right)\left[({a_0}^2 {b_1}^2 - {a_1}^2 {b_0}^2)(b_0 + a_0) + 2a_0 b_0(a_1 {b_0}^2 + a_1 a_0 b_0 - b_1 a_0 b_0 - {a_0}^2 b_1)\right]$$

$$= \frac{1}{4}\left(\frac{1}{b_1 - a_1}\right)\left(\frac{1}{{a_0}^2 {b_0}^2}\right)\left[({a_0}^2 {b_1}^2 - {a_1}^2 {b_0}^2)(b_0 + a_0) + 2a_0 b_0(a_1 {b_0}^2 - {a_0}^2 b_1) + 2{a_0}^2 {b_0}^2 (a_1 - b_1)\right] \quad (37)$$

For $\frac{b_1}{b_0} - 1 < i < \frac{b_1}{a_0} - 1$ region,

$$E_3(i) = \int_{\frac{b_1}{b_0}-1}^{\frac{b_1}{a_0}-1} \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{{b_1}^2}{(i+1)^2} - {a_0}^2\right) i \, di$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[{b_1}^2\left(\frac{1}{1+i} + \ln(1+i)\right) - {a_0}^2 \frac{i^2}{2}\right]_{\frac{b_1}{b_0}-1}^{\frac{b_1}{a_0}-1}$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[b_1{}^2 \left(\frac{1}{1 + \frac{b_1}{a_0} - 1} + \ln\left(1 + \frac{b_1}{a_0} - 1\right) \right) - a_0{}^2 \frac{\left(\frac{b_1}{a_0} - 1\right)^2}{2} \right.$$

$$\left. - b_1{}^2 \left(\frac{1}{1 + \frac{b_1}{b_0} - 1} + \ln\left(1 + \frac{b_1}{b_0} - 1\right) \right) + a_0{}^2 \frac{\left(\frac{b_1}{b_0} - 1\right)^2}{2} \right]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[b_1{}^2 \left(\frac{a_0}{b_1} + \ln\left(\frac{b_1}{a_0}\right) - \frac{b_0}{b_1} - \ln\left(\frac{b_1}{b_0}\right) \right) + a_0{}^2 \left(\frac{(b_1 - b_0)^2 a_0{}^2}{2a_0{}^2 b_0{}^2} - \frac{(b_1 - a_0)^2 b_0{}^2}{2a_0{}^2 b_0{}^2} \right) \right]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(b_1{}^2 \left[\frac{a_0 - b_0}{b_1} + ln\left(\frac{\frac{b_1}{a_0}}{\frac{b_1}{b_0}} \right) \right] \right.$$

$$\left. + \frac{a_0{}^2}{2a_0{}^2 b_0{}^2}\left[(b_1{}^2 - 2b_1 b_0 + b_0{}^2) a_0{}^2 - (b_1{}^2 - 2b_1 a_0 + a_0{}^2) b_0{}^2 \right] \right)$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(b_1(a_0 - b_0) + b_1{}^2 ln\left(\frac{b_0}{a_0}\right) \right.$$

$$\left. + \frac{1}{2b_0{}^2}\left[b_1{}^2 a_0{}^2 - 2b_1 b_0 a_0{}^2 + b_0{}^2 a_0{}^2 - b_1{}^2 b_0{}^2 + 2b_1 a_0 b_0{}^2 - a_0{}^2 b_0{}^2 \right] \right)$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(b_1(a_0 - b_0) + b_1{}^2 ln\left(\frac{b_0}{a_0}\right) \right. \tag{38}$$

$$\left. + \frac{1}{2b_0{}^2}\left[b_1{}^2 (a_0{}^2 - b_0{}^2) - 2b_1 b_0 a_0 (a_0 - b_0) \right] \right)$$

Equations 36 and 38, $E_1(i)$ and $E_3(i)$ respectively, are summed up first for the ease of computation.

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{a_1{}^2\left(b_0{}^2 - a_0{}^2\right)}{2a_0{}^2} - \frac{2b_0 a_0 (b_0 - a_0) a_1}{2a_0{}^2} + a_1(b_0 - a_0) + a_1{}^2 ln\left(\frac{a_0}{b_0}\right)\right.$$

$$\left. - \frac{b_1{}^2\left(b_0{}^2 - a_0{}^2\right)}{2b_0{}^2} + \frac{2b_0 a_0 (b_0 - a_0) b_1}{2b_0{}^2} - b_1(b_0 - a_0) - b_1{}^2 ln\left(\frac{a_0}{b_0}\right)\right)$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\left(b_0{}^2 - a_0{}^2\right)\left(\frac{a_1{}^2}{2a_0{}^2} - \frac{b_1{}^2}{2b_0{}^2}\right) + 2b_0 a_0 (b_0 - a_0)(\frac{b_1}{2b_0{}^2} - \frac{a_1}{2a_0{}^2})\right.$$

$$\left. + (b_0 - a_0)(a_1 - b_1) + ln\left(\frac{a_0}{b_0}\right)(a_1{}^2 - b_1{}^2)\right)$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\left(a_1{}^2 b_0{}^2 - b_1{}^2 a_0{}^2\right)\left(\frac{\left(b_0{}^2 - a_0{}^2\right)}{2a_0{}^2 b_0{}^2}\right) + \left(\frac{2b_0 a_0}{2a_0{}^2 b_0{}^2}\right)(b_0 - a_0)\left(b_1 a_0{}^2 - a_1 b_0{}^2\right)\right.$$

$$\left. + \left(\frac{2a_0{}^2 b_0{}^2}{2a_0{}^2 b_0{}^2}\right)(b_0 - a_0)(a_1 - b_1) + \left(\frac{2a_0{}^2 b_0{}^2}{2a_0{}^2 b_0{}^2}\right) ln\left(\frac{a_0}{b_0}\right)(a_1{}^2 - b_1{}^2)\right)$$

$$= \frac{1}{4a_0{}^2 b_0{}^2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\left(b_0{}^2 - a_0{}^2\right)\left(a_1{}^2 b_0{}^2 - b_1{}^2 a_0{}^2\right) + 2b_0 a_0 (b_0 - a_0)\left(b_1 a_0{}^2 - a_1 b_0{}^2\right)\right.$$

$$\left. + 2a_0{}^2 b_0{}^2 (b_0 - a_0)(a_1 - b_1) + 2a_0{}^2 b_0{}^2 ln\left(\frac{a_0}{b_0}\right)(a_1{}^2 - b_1{}^2)\right)$$

$$= \frac{1}{4a_0{}^2 b_0{}^2}\frac{1}{b_1 - a_1}\Big((b_0 + a_0)\left(a_1{}^2 b_0{}^2 - b_1{}^2 a_0{}^2\right) + 2b_0 a_0\left(b_1 a_0{}^2 - a_1 b_0{}^2\right)$$
$$+ 2a_0{}^2 b_0{}^2 (a_1 - b_1)\Big) - \frac{1}{2}\frac{1}{b_0 - a_0} ln\left(\frac{a_0}{b_0}\right)(a_1 + b_1) \qquad (39)$$

The sum of equations 37 and 39 gives the sum of $E_1(i)$, $E_2(i)$ and $E_3(i)$ and the overall expected value for the problem.

$$= \frac{1}{4{a_0}^2{b_0}^2}\frac{1}{b_1 - a_1}\left[(b_0 + a_0)\left({a_1}^2{b_0}^2 - {b_1}^2{a_0}^2\right) + 2b_0a_0\left(b_1{a_0}^2 - a_1{b_0}^2\right) + 2{a_0}^2{b_0}^2(a_1 - b_1)\right.$$

$$\left. - (b_0 + a_0)\left({a_1}^2{b_0}^2 - {b_1}^2{a_0}^2\right) - 2a_0b_0\left(b_1{a_0}^2 - a_1{b_0}^2\right) + 2{a_0}^2{b_0}^2(a_1 - b_1)\right]$$

$$- \frac{1}{2}\frac{1}{b_0 - a_0} ln\left(\frac{a_0}{b_0}\right)(a_1 + b_1)$$

$$= \frac{1}{4{a_0}^2{b_0}^2}\frac{1}{b_1 - a_1}\left[4{a_0}^2{b_0}^2(a_1 - b_1)\right] - \frac{1}{2}\frac{(a_1 + b_1)}{b_0 - a_0} ln\left(\frac{a_0}{b_0}\right)$$

$$\boldsymbol{E(i) = -1 - \frac{1}{2}\frac{(a_1 + b_1)}{b_0 - a_0} ln\left(\frac{a_0}{b_0}\right)} \qquad (40)$$

3.2.4. The Variance

The following indefinite integral is used in the derivation of the variance.

$$\int \frac{i^2}{(1 + i)^2} di = \frac{i^2 - 2}{1 + i} - 2\ln(1 + i)$$

Variance is calculated by equation 41 below:

$$\sigma^2 = \int f(i)\, i^2\, di - [E(i)]^2 \qquad (41)$$

The integration should be made for all the regions possible. In this problem three different regions are integrated whose results are shown in equations 42, 43 and 44 respectively.

For $\frac{a_1}{b_0} - 1 < i < \frac{a_1}{a_0} - 1$ region,

$$\int_{\frac{a_1}{b_0}-1}^{\frac{a_1}{a_0}-1} \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left({b_0}^2 - \frac{{a_1}^2}{(i + 1)^2}\right) i^2\, di$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[{b_0}^2\frac{i^3}{3} - {a_1}^2\left(\frac{i^2 - 2}{1 + i} - 2\ln(1 + i)\right)\right]_{\frac{a_1}{b_0}-1}^{\frac{a_1}{a_0}-1}$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[{b_0}^2 \frac{(a_1 - a_0)^3}{3{a_0}^3} - {a_1}^2 \left(\frac{\left(\frac{a_1}{a_0} - 1\right)^2 - 2}{1 + \frac{a_1}{a_0} - 1} - 2\ln\left(1 + \frac{a_1}{a_0} - 1\right) \right) - {b_0}^2 \frac{(a_1 - b_0)^3}{3{b_0}^3} \right.$$

$$\left. + {a_1}^2 \left(\frac{\left(\frac{a_1}{b_0} - 1\right)^2 - 2}{1 + \frac{a_1}{b_0} - 1} - 2\ln\left(1 + \frac{a_1}{b_0} - 1\right) \right) \right]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[\frac{{b_0}^2}{3}\left(\frac{(a_1 - a_0)^3}{{a_0}^3} - \frac{(a_1 - b_0)^3}{{b_0}^3} \right) \right.$$

$$\left. + {a_1}^2 \left(\frac{\frac{(a_1 - b_0)^2 - 2{b_0}^2}{{b_0}^2}}{\frac{a_1}{b_0}} - \frac{\frac{(a_1 - a_0)^2 - 2{a_0}^2}{{a_0}^2}}{\frac{a_1}{a_0}} \right) + 2{a_1}^2 \left(\ln\left(\frac{a_1}{a_0}\right) - \ln\left(\frac{a_1}{b_0}\right) \right) \right]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[\left(\frac{{b_0}^2}{3{a_0}^3{b_0}^3}\left({b_0}^3({a_1}^3 - 3{a_1}^2 a_0 + 3a_1{a_0}^2 - {a_0}^3) \right.\right.\right.$$

$$\left.\left. - {a_0}^3({a_1}^3 - 3{a_1}^2 b_0 + 3a_1{b_0}^2 - {b_0}^3) \right) \right)$$

$$\left. + a_1 \left(\frac{{a_1}^2 - 2a_1 b_0 + {b_0}^2 - 2{b_0}^2}{b_0} - \frac{{a_1}^2 - 2a_1 a_0 + {a_0}^2 - 2{a_0}^2}{a_0} \right) + 2{a_1}^2 \ln\left(\frac{b_0}{a_0}\right) \right] =$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[\left(\frac{1}{3{a_0}^3 b_0}\left({b_0}^3{a_1}^3 - 3{b_0}^3{a_1}^2 a_0 + 3{b_0}^3 a_1{a_0}^2 - {b_0}^3{a_0}^3 - {a_0}^3{a_1}^3 \right.\right.\right.$$

$$\left.\left. + 3{a_0}^3{a_1}^2 b_0 - 3{a_0}^3 a_1{b_0}^2 + {a_0}^3{b_0}^3 \right) \right)$$

$$\left. + \frac{a_1}{a_0 b_0}\left(a_0({a_1}^2 - 2a_1 b_0 - {b_0}^2) - b_0({a_1}^2 - 2a_1 a_0 - {a_0}^2) \right) + 2{a_1}^2 \ln\left(\frac{b_0}{a_0}\right) \right]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[\left(\frac{1}{3a_0{}^3b_0}\left(a_1{}^3\left(b_0{}^3 - a_0{}^3\right) - 3b_0a_1{}^2a_0\left(b_0{}^2 - a_0{}^2\right) + 3b_0{}^2a_1a_0{}^2(b_0 - a_0)\right)\right)\right.$$

$$\left. + \frac{a_1}{a_0b_0}\left(a_0a_1{}^2 - 2a_0a_1b_0 - a_0b_0{}^2 - b_0a_1{}^2 + 2b_0a_1a_0 + b_0a_0{}^2\right) + 2a_1{}^2\ln\left(\frac{b_0}{a_0}\right)\right]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[\left(\frac{a_1}{3a_0{}^3b_0}\left(a_1{}^2\left(b_0{}^3 - a_0{}^3\right) - 3b_0a_1a_0\left(b_0{}^2 - a_0{}^2\right)\right.\right.\right.$$
$$\left.\left.\left. + 3b_0{}^2a_0{}^2(b_0 - a_0)\right)\right) + \frac{a_1}{a_0b_0}\left(a_0(a_1{}^2 - b_0{}^2) - b_0(a_1{}^2 - a_0{}^2)\right) + 2a_1{}^2\ln\left(\frac{b_0}{a_0}\right)\right] \tag{42}$$

For $\frac{a_1}{a_0} - 1 < i < \frac{b_1}{b_0} - 1$ region,

$$\int_{\frac{a_1}{a_0}-1}^{\frac{b_1}{b_0}-1}\frac{1}{2}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)i^2\,di = \frac{1}{2}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left[\frac{i^3}{3}\right]_{\frac{a_1}{a_0}-1}^{\frac{b_1}{b_0}-1} = \frac{1}{2}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left[\frac{\left(\frac{b_1}{b_0} - 1\right)^3}{3} - \frac{\left(\frac{a_1}{a_0} - 1\right)^3}{3}\right]$$

$$= \frac{1}{6}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left[\frac{(b_1 - b_0)^3a_0{}^3}{a_0{}^3b_0{}^3} - \frac{(a_1 - a_0)^3b_0{}^3}{a_0{}^3b_0{}^3}\right]$$

$$= \frac{1}{6a_0{}^3b_0{}^3}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left[\left(b_1{}^3 - 3b_1{}^2b_0 + 3b_1b_0{}^2 - b_0{}^3\right)a_0{}^3 - (a_1{}^3 - 3a_1{}^2a_0 + 3a_1a_0{}^2 - a_0{}^3)b_0{}^3\right]$$

$$= \frac{1}{6a_0{}^3b_0{}^3}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left[b_1{}^3a_0{}^3 - 3a_0{}^3b_1{}^2b_0 + 3a_0{}^3b_1b_0{}^2 - a_0{}^3b_0{}^3 - a_1{}^3b_0{}^3 + 3b_0{}^3a_1{}^2a_0\right.$$
$$\left. - 3b_0{}^3a_1a_0{}^2 + b_0{}^3a_0{}^3\right]$$

$$= \frac{1}{6a_0{}^3b_0{}^3}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left[b_1{}^3a_0{}^3 - a_1{}^3b_0{}^3 + 3a_0b_0(b_0{}^2a_1{}^2 - a_0{}^2b_1{}^2)\right.$$
$$\left. + 3a_0{}^2b_0{}^2(a_0b_1 - b_0a_1)\right] \tag{43}$$

For $\frac{b_1}{b_0} - 1 < i < \frac{b_1}{a_0} - 1$ region,

$$\int_{\frac{b_1}{b_0}-1}^{\frac{b_1}{a_0}-1} \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{{b_1}^2}{(i+1)^2} - {a_0}^2\right) i^2\, di$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[{b_1}^2\left(\frac{i^2 - 2}{1 + i} - 2\ln(1 + i)\right) - {a_0}^2\frac{i^3}{3}\right]_{\frac{b_1}{b_0}-1}^{\frac{b_1}{a_0}-1}$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[{b_1}^2\left(\frac{\left(\frac{b_1}{a_0} - 1\right)^2 - 2}{1 + \frac{b_1}{a_0} - 1} - 2\ln\left(1 + \frac{b_1}{a_0} - 1\right)\right) - {a_0}^2\frac{(b_1 - a_0)^3}{3{a_0}^3}\right.$$

$$\left. - {b_1}^2\left(\frac{\left(\frac{b_1}{b_0} - 1\right)^2 - 2}{1 + \frac{b_1}{b_0} - 1} - 2\ln\left(1 + \frac{b_1}{b_0} - 1\right)\right) + {a_0}^2\frac{(b_1 - b_0)^3}{3{b_0}^3}\right]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[{b_1}^2\left(\frac{(b_1 - a_0)^2 - 2{a_0}^2}{a_0 b_1} - \frac{(b_1 - b_0)^2 - 2{b_0}^2}{b_0 b_1}\right)\right.$$

$$\left. + \frac{{a_0}^2}{3}\left(\frac{(b_1 - b_0)^3}{{b_0}^3} - \frac{(b_1 - a_0)^3}{{a_0}^3}\right) + 2{b_1}^2\left(\ln\left(\frac{b_1}{b_0}\right) - \ln\left(\frac{b_1}{a_0}\right)\right)\right]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[\frac{b_1}{b_0 a_0}\left(b_0({b_1}^2 - 2b_1 a_0 + {a_0}^2 - 2{a_0}^2) - a_0({b_1}^2 - 2b_1 b_0 + {b_0}^2 - 2{b_0}^2)\right)\right.$$

$$+ \left(\frac{{a_0}^2}{3{a_0}^3{b_0}^3}\left({a_0}^3\left({b_1}^3 - 3{b_1}^2 b_0 + 3b_1{b_0}^2 - {b_0}^3\right) - {b_0}^3\left({b_1}^3 - 3{b_1}^2 a_0 + 3b_1{a_0}^2 - {a_0}^3\right)\right)\right)$$

$$\left. + 2{b_1}^2\ln\left(\frac{a_0}{b_0}\right)\right]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\Bigg[\frac{b_1}{b_0 a_0}\left(b_0 {b_1}^2 - 2b_0 b_1 a_0 - b_0 {a_0}^2 - a_0 {b_1}^2 + 2a_0 b_1 b_0 + a_0 {b_0}^2\right)$$

$$+ \left(\frac{1}{3{b_0}^3 a_0}\left({a_0}^3{b_1}^3 - 3{a_0}^3{b_1}^2 b_0 + 3{a_0}^3 b_1 {b_0}^2 - {b_0}^3{a_0}^3 - {b_0}^3{b_1}^3 + 3{b_0}^3{b_1}^2 a_0 - 3{b_0}^3 b_1 {a_0}^2\right.\right.$$

$$\left.\left. + {a_0}^3{b_0}^3\right)\right) + 2{b_1}^2 \ln\left(\frac{a_0}{b_0}\right)\Bigg]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\Bigg[\frac{b_1}{a_0 b_0}\left(b_0({b_1}^2 - {a_0}^2) - a_0({b_1}^2 - {b_0}^2)\right)$$

$$+ \left(\frac{1}{3{b_0}^3 a_0}\left({b_1}^3\left({a_0}^3 - {b_0}^3\right) - 3b_0{b_1}^2 a_0\left({a_0}^2 - {b_0}^2\right) + 3{b_0}^2 b_1 {a_0}^2(a_0 - b_0)\right)\right) + 2{b_1}^2 \ln\left(\frac{a_0}{b_0}\right)\Bigg]$$

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\Bigg[\frac{b_1}{a_0 b_0}\left(b_0({b_1}^2 - {a_0}^2) - a_0({b_1}^2 - {b_0}^2)\right)$$

$$+ \left(\frac{b_1}{3{b_0}^3 a_0}\left({b_1}^2\left({a_0}^3 - {b_0}^3\right) - 3b_0 b_1 a_0\left({a_0}^2 - {b_0}^2\right) + 3{b_0}^2{a_0}^2(a_0 - b_0)\right)\right) \tag{44}$$

$$+ 2{b_1}^2 \ln\left(\frac{a_0}{b_0}\right)\Bigg]$$

The overall integration is complete with the sum of equations 42, 43, and 44. The result is shown in equation 45.

$$= \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[\left(\frac{a_1}{3{a_0}^3 b_0}\left({a_1}^2\left({b_0}^3 - {a_0}^3\right) - 3b_0 a_1 a_0\left({b_0}^2 - {a_0}^2\right) + 3{b_0}^2{a_0}^2(b_0 - a_0)\right)\right)\right.$$

$$\left. + \left(\frac{b_1}{3{b_0}^3 a_0}\left({b_1}^2\left({a_0}^3 - {b_0}^3\right) - 3b_0 b_1 a_0\left({a_0}^2 - {b_0}^2\right) + 3{b_0}^2{a_0}^2(a_0 - b_0)\right)\right)\right]$$

$$+ \frac{1}{6{a_0}^3{b_0}^3}\left(\frac{b_0 + a_0}{b_1 - a_1}\right)\left[{b_1}^3{a_0}^3 - {a_1}^3{b_0}^3 + 3a_0 b_0({b_0}^2{a_1}^2 - {a_0}^2{b_1}^2) + 3{a_0}^2{b_0}^2(a_0 b_1 - b_0 a_1)\right]$$

$$+ \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[\frac{a_1}{a_0 b_0}\left(a_0({a_1}^2 - {b_0}^2) - b_0({a_1}^2 - {a_0}^2)\right)\right.$$

$$\left. + \frac{b_1}{a_0 b_0}\left(b_0({b_1}^2 - {a_0}^2) - a_0({b_1}^2 - {b_0}^2)\right)\right] + \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(2{a_1}^2\ln\left(\frac{b_0}{a_0}\right) + 2{b_1}^2\ln\left(\frac{a_0}{b_0}\right)\right)$$

$$= \frac{1}{6a_0 b_0}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[\left(\frac{a_1}{{a_0}^2}\left({a_1}^2\left({b_0}^3 - {a_0}^3\right) - 3b_0 a_1 a_0\left({b_0}^2 - {a_0}^2\right) + 3{b_0}^2{a_0}^2(b_0 - a_0)\right)\right)\right.$$

$$\left. + \left(\frac{b_1}{{b_0}^2}\left({b_1}^2\left({a_0}^3 - {b_0}^3\right) - 3b_0 b_1 a_0\left({a_0}^2 - {b_0}^2\right) + 3{b_0}^2{a_0}^2(a_0 - b_0)\right)\right)\right]$$

$$+ \frac{1}{6{a_0}^3{b_0}^3}\left(\frac{1}{b_1 - a_1}\right)\left(\frac{{b_0}^2 - {a_0}^2}{b_0 - a_0}\right)\left[{b_1}^3{a_0}^3 - {a_1}^3{b_0}^3 + 3a_0 b_0({b_0}^2{a_1}^2 - {a_0}^2{b_1}^2) + 3{a_0}^2{b_0}^2(a_0 b_1\right.$$

$$\left. - b_0 a_1)\right]$$

$$+ \frac{1}{2a_0 b_0}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left[a_1\left(a_0({a_1}^2 - {b_0}^2) - b_0({a_1}^2 - {a_0}^2)\right)\right.$$

$$\left. + b_1\left(b_0({b_1}^2 - {a_0}^2) - a_0({b_1}^2 - {b_0}^2)\right)\right] + \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(2{a_1}^2\ln\left(\frac{b_0}{a_0}\right) - 2{b_1}^2\ln\left(\frac{b_0}{a_0}\right)\right)$$

$$= \frac{1}{6a_0 b_0}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\left({b_0}^3 - {a_0}^3\right)\left(\frac{{a_1}^3}{{a_0}^2} - \frac{{b_1}^3}{{b_0}^2}\right) - 3b_0 a_0\left({b_0}^2 - {a_0}^2\right)\left(\frac{{a_1}^2}{{a_0}^2} - \frac{{b_1}^2}{{b_0}^2}\right)\right.$$

$$\left. + 3{b_0}^2{a_0}^2(b_0 - a_0)\left(\frac{a_1}{{a_0}^2} - \frac{b_1}{{b_0}^2}\right)\right) + \frac{1}{6{a_0}^3{b_0}^3}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}({b_0}^2$$

$$- {a_0}^2)\left[{b_1}^3{a_0}^3 - {a_1}^3{b_0}^3 + 3a_0 b_0({b_0}^2{a_1}^2 - {a_0}^2{b_1}^2) + 3{a_0}^2{b_0}^2(a_0 b_1 - b_0 a_1)\right]$$

$$+\frac{1}{2a_0b_0}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}\left[a_0{a_1}^3-a_0a_1{b_0}^2-b_0{a_1}^3+b_0a_1{a_0}^2+b_0{b_1}^3-b_0b_1{a_0}^2-a_0{b_1}^3\right.$$

$$\left.+a_0b_1{b_0}^2\right]+\frac{1}{2}\frac{1}{b_0-a_0}\frac{2}{b_1-a_1}\left(({a_1}^2-{b_1}^2)\ln\left(\frac{b_0}{a_0}\right)\right)$$

$$=\frac{1}{6{a_0}^3{b_0}^3}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}\Big(({b_0}^3-{a_0}^3)({a_1}^3{b_0}^2-{b_1}^3{a_0}^2)$$

$$-3b_0a_0({b_0}^2-{a_0}^2)({a_1}^2{b_0}^2-{b_1}^2{a_0}^2)+3{b_0}^2{a_0}^2(b_0-a_0)(a_1{b_0}^2-b_1{a_0}^2)+({b_0}^2$$

$$-{a_0}^2)\left[{b_1}^3{a_0}^3-{a_1}^3{b_0}^3+3a_0b_0({b_0}^2{a_1}^2-{a_0}^2{b_1}^2)+3{a_0}^2{b_0}^2(a_0b_1-b_0a_1)\right]\Big)$$

$$+\frac{1}{2a_0b_0}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}\left[a_0{a_1}^3-a_0a_1{b_0}^2-b_0{a_1}^3+b_0a_1{a_0}^2+b_0{b_1}^3-b_0b_1{a_0}^2\right.$$

$$\left.-a_0{b_1}^3+a_0b_1{b_0}^2\right]+\frac{b_1+a_1}{b_0-a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$=\frac{1}{6{a_0}^3{b_0}^3}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}\Big(({b_0}^3-{a_0}^3)({a_1}^3{b_0}^2-{b_1}^3{a_0}^2)$$

$$-3b_0a_0({b_0}^2-{a_0}^2)({a_1}^2{b_0}^2-{b_1}^2{a_0}^2)+3{b_0}^2{a_0}^2(b_0-a_0)(a_1{b_0}^2-b_1{a_0}^2)$$

$$+({b_0}^2-{a_0}^2)({b_1}^3{a_0}^3-{a_1}^3{b_0}^3)+3a_0b_0({b_0}^2-{a_0}^2)({b_0}^2{a_1}^2-{a_0}^2{b_1}^2)$$

$$+3{a_0}^2{b_0}^2({b_0}^2-{a_0}^2)(a_0b_1-b_0a_1)\Big)$$

$$+\frac{1}{2a_0b_0}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}\left[({a_1}^3-{b_1}^3)(a_0-b_0)+a_0b_0(a_1-b_1)(a_0-b_0)\right]$$

$$+\frac{b_1+a_1}{b_0-a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$= \frac{1}{6{a_0}^3{b_0}^3}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\Big(({b_0}^3 - {a_0}^3)({a_1}^3{b_0}^2 - {b_1}^3{a_0}^2)$$

$$+ 3{b_0}^2{a_0}^2(b_0 - a_0)(a_1{b_0}^2 - b_1{a_0}^2) + ({b_0}^2 - {a_0}^2)({b_1}^3{a_0}^3 - {a_1}^3{b_0}^3)$$

$$+ 3{a_0}^2{b_0}^2({b_0}^2 - {a_0}^2)(a_0b_1 - b_0a_1)\Big)$$

$$+ \frac{1}{2a_0b_0}\frac{1}{b_0 - a_0}\frac{(a_0 - b_0)}{b_1 - a_1}\left[({a_1}^3 - {b_1}^3) + a_0b_0(a_1 - b_1)\right] + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$= \frac{1}{6{a_0}^3{b_0}^3}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\Big({b_0}^5{a_1}^3 - {a_0}^3{a_1}^3{b_0}^2 - {b_0}^3{b_1}^3{a_0}^2 + {a_0}^5{b_1}^3 + {b_0}^2{b_1}^3{a_0}^3$$

$$- {a_0}^5{b_1}^3 - {b_0}^5{a_1}^3 + {a_0}^2{a_1}^3{b_0}^3$$

$$+ 3{b_0}^2{a_0}^2\big(b_0a_1{b_0}^2 - a_0a_1{b_0}^2 - b_0b_1{a_0}^2 + a_0b_1{a_0}^2 + {b_0}^2a_0b_1 - {b_0}^2b_0a_1 - {a_0}^2a_0b_1$$

$$+ {a_0}^2b_0a_1\big)\Big) - \frac{1}{2a_0b_0}\frac{1}{b_1 - a_1}\left[({a_1}^3 - {b_1}^3) + a_0b_0(a_1 - b_1)\right] + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$= \frac{1}{6{a_0}^3{b_0}^3}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\Big(-{a_0}^3{a_1}^3{b_0}^2 - {b_0}^3{b_1}^3{a_0}^2 + {b_0}^2{b_1}^3{a_0}^3 + {a_0}^2{a_1}^3{b_0}^3$$

$$+ 3{b_0}^2{a_0}^2\big(-a_0a_1{b_0}^2 - b_0b_1{a_0}^2 + {b_0}^2a_0b_1 + {a_0}^2b_0a_1\big)\Big)$$

$$- \frac{1}{2a_0b_0}\frac{1}{b_1 - a_1}\left[({a_1}^3 - {b_1}^3) + a_0b_0(a_1 - b_1)\right] + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$= \frac{1}{6{a_0}^3{b_0}^3}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\Big(({a_1}^3 - {b_1}^3){a_0}^2{b_0}^2(b_0 - a_0) + 3{b_0}^3{a_0}^3(b_0 - a_0)(b_1 - a_1)\Big)$$

$$- \frac{1}{2a_0b_0}\frac{1}{b_1 - a_1}\left[({a_1}^3 - {b_1}^3) + a_0b_0(a_1 - b_1)\right] + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$= \frac{1}{6{a_0}^3{b_0}^3}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left({a_1}^3 - {b_1}^3\right){a_0}^2{b_0}^2(b_0 - a_0)$$

$$+ \frac{1}{6{a_0}^3{b_0}^3}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}3{b_0}^3{a_0}^3(b_0 - a_0)(b_1 - a_1) - \frac{1}{2a_0b_0}\frac{1}{b_1 - a_1}\left({a_1}^3 - {b_1}^3\right)$$

$$- \frac{1}{2a_0b_0}\frac{1}{b_1 - a_1}a_0b_0(a_1 - b_1) + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$= \frac{1}{6a_0b_0}\frac{1}{b_1 - a_1}\left({a_1}^3 - {b_1}^3\right) + \frac{1}{2} - \frac{1}{2a_0b_0}\frac{1}{b_1 - a_1}\left({a_1}^3 - {b_1}^3\right) + \frac{1}{2} + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$= -\frac{1}{3a_0b_0}\frac{1}{b_1 - a_1}\left({a_1}^3 - {b_1}^3\right) + 1 + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$= \frac{1}{3a_0b_0}\frac{1}{b_1 - a_1}\left({b_1}^3 - {a_1}^3\right) + 1 + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$= \frac{1}{3a_0b_0}\left((b_1 - a_1)^2 + 3a_1b_1\right) + 1 + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right) \tag{45}$$

The variance is obtained by subtracting square of equation 40 – the square of the expected value – from the integration from equation 45 as the definition of the variance requires. Equation 46 shows the variance expression.

$$\sigma^2 = \frac{1}{3a_0b_0}\left((b_1 - a_1)^2 + 3a_1b_1\right) + 1 + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right) - \left(-1 - \frac{1}{2}\frac{(a_1 + b_1)}{b_0 - a_0}\,ln\left(\frac{a_0}{b_0}\right)\right)^2$$

$$= \frac{\left({b_1}^2 + a_1b_1 + {a_1}^2\right)}{3a_0b_0} + 1 + \frac{b_1 + a_1}{b_0 - a_0}\ln\left(\frac{a_0}{b_0}\right)$$

$$- \left[1 + \frac{(a_1 + b_1)}{b_0 - a_0}\,ln\left(\frac{a_0}{b_0}\right) + \frac{1}{4}\left(\frac{(a_1 + b_1)}{b_0 - a_0}\right)^2 ln^2\left(\frac{a_0}{b_0}\right)\right]$$

$$= \frac{\left(b_1{}^2 + a_1 b_1 + a_1{}^2\right)}{3a_0 b_0} + 1 + \frac{b_1 + a_1}{b_0 - a_0} \ln\left(\frac{a_0}{b_0}\right) - 1 - \frac{(a_1 + b_1)}{b_0 - a_0} ln\left(\frac{a_0}{b_0}\right)$$

$$- \frac{1}{4}\left(\frac{(a_1 + b_1)}{b_0 - a_0}\right)^2 ln^2\left(\frac{a_0}{b_0}\right)$$

$$\sigma^2 = \frac{\left(\boldsymbol{b_1}^{\mathbf{2}} + \boldsymbol{a_1 b_1} + \boldsymbol{a_1}^{\mathbf{2}}\right)}{\mathbf{3}\boldsymbol{a_0 b_0}} - \frac{\mathbf{1}}{\mathbf{4}}\left(\frac{(\boldsymbol{a_1} + \boldsymbol{b_1})}{\boldsymbol{b_0} - \boldsymbol{a_0}}\right)^{\mathbf{2}} \boldsymbol{ln}^{\mathbf{2}}\left(\frac{\boldsymbol{a_0}}{\boldsymbol{b_0}}\right) \qquad (46)$$

3.3 Two Period IRR Problem: A_0 Constant, A_1 Constant, A_2 Uniform

Terms A_0 and A_1 are assumed to be deterministic values. The A_2 is uniformly distributed between $[a_2, b_2]$ with the distribution function $f(A_2) = 1/(b_2 - a_2)$

For this problem the meaningful root is as follows: $i = \frac{-A_1 + 2A_0 - \sqrt{A_1{}^2 + 4A_0 A_2}}{-2A_0}$

3.3.1. Derivation of the PDF with the Distribution Function Method

1. Step :

$$P(Z \le z) = P\left(\sqrt{A + Bx} \le z\right) = P(A + Bx \le z^2) = P\left(x \le \frac{z^2 - A}{B}\right)$$

$$= F_x\left(\frac{z^2 - A}{B}\right) \textit{ where } A: \mathrm{A}_1^2 \text{ and } \mathrm{B}: 4\mathrm{A}_0 = F_x\left(\frac{z^2 - \mathrm{A}_1^2}{4\mathrm{A}_0}\right)$$

$$\text{for uniform distribution } F_x = \frac{x - a_2}{b_2 - a_2}$$

$$= F_x\left(\frac{z^2 - \mathrm{A}_1^2}{4\mathrm{A}_0}\right) = \frac{\frac{z^2 - \mathrm{A}_1^2}{4\mathrm{A}_0} - a_2}{b_2 - a_2}$$

$$F(Z) = \frac{z^2 - A_1^2 - 4A_0 a_2}{4A_0(b_2 - a_2)}$$

where limits are: $a_2 \le \frac{z^2 - A_1^2}{4A_0} \le b_2$, thus $\sqrt{A_1^2 + 4A_0 a_2} \le z \le \sqrt{A_1^2 + 4A_0 b_2}$

2. Step

$$P\left(\frac{x}{A} \le z\right) = P(x \le zA) = F_x(2A_0 z)$$

$$F(Z) = \frac{4{A_0}^2 z^2 - A_1^2 - 4A_0 a_2}{4A_0(b_2 - a_2)}$$

$$limits\ are\ \frac{\sqrt{A_1^2 + 4A_0 a_2}}{2A_0} \le z \le \frac{\sqrt{A_1^2 + 4A_0 b_2}}{2A_0}$$

3. Step

$$P(x + A \le i) = P(x \le i - A) = F_x\left(i - \frac{A_1}{2A_0} + 1\right)$$

$$F(i) = \frac{4{A_0}^2(i - \frac{A_1}{2A_0} + 1)^2 - A_1^2 - 4A_0 a_2}{4A_0(b_2 - a_2)} \tag{47}$$

where the limits are $\frac{\sqrt{A_1^2 + 4A_0 a_2}}{2A_0} + \frac{A_1}{2A_0} - 1 \le i \le \frac{\sqrt{A_1^2 + 4A_0 b_2}}{2A_0} + \frac{A_1}{2A_0} - 1$

The CDF is checked for validity:

$$\frac{4{A_0}^2(i - \frac{A_1}{2A_0} + 1)^2 - A_1^2 - 4A_0 a_2}{4A_0(b_2 - a_2)}\Bigg|_{\frac{\sqrt{A_1^2+4A_0 a_2}}{2A_0}+\frac{A_1}{2A_0}-1}^{\frac{\sqrt{A_1^2+4A_0 b_2}}{2A_0}+\frac{A_1}{2A_0}-1} =$$

$$= \frac{4{A_0}^2\left(\frac{\sqrt{A_1^2+4A_0 b_2}}{2A_0}+\frac{A_1}{2A_0}-1-\frac{A_1}{2A_0}+1\right)^2-A_1^2-4A_0 a_2}{4A_0(b_2-a_2)}$$

$$- \frac{4{A_0}^2(\frac{\sqrt{A_1^2+4A_0 a_2}}{2A_0}+\frac{A_1}{2A_0}-1-\frac{A_1}{2A_0}+1)^2-A_1^2-4A_0 a_2}{4A_0(b_2-a_2)}$$

$$= \frac{4{A_0}^2\left(\frac{\sqrt{A_1^2+4A_0 b_2}}{2A_0}\right)^2-A_1^2-4A_0 a_2}{4A_0(b_2-a_2)} - \frac{4{A_0}^2\left(\frac{\sqrt{A_1^2+4A_0 a_2}}{2A_0}\right)^2-A_1^2-4A_0 a_2}{4A_0(b_2-a_2)}$$

$$= \frac{4{A_0}^2\left(\frac{A_1^2+4A_0 b_2}{4{A_0}^2}\right)-A_1^2-4A_0 a_2}{4A_0(b_2-a_2)} - \frac{4{A_0}^2\left(\frac{A_1^2+4A_0 a_2}{4{A_0}^2}\right)-A_1^2-4A_0 a_2}{4A_0(b_2-a_2)}$$

$$= \frac{A_1^2+4A_0 b_2-A_1^2-4A_0 a_2}{4A_0(b_2-a_2)} - \frac{A_1^2+4A_0 a_2-A_1^2-4A_0 a_2}{4A_0(b_2-a_2)}$$

$$\frac{4A_0 b_2-4A_0 a_2}{4A_0(b_2-a_2)} = \frac{4A_0(b_2-a_2)}{4A_0(b_2-a_2)} = 1$$

The PDF is found next:

$$f(i) = \frac{dF(i)}{di} = \frac{4{A_0}^2 2(i-\frac{A_1}{2A_0}+1)}{4A_0(b_2-a_2)} = \frac{2A_0}{(b_2-a_2)}(i-\frac{A_1}{2A_0}+1)$$

$$f(i) = \frac{2A_0}{b_2-a_2}(i-\frac{A_1}{2A_0}+1) \tag{48}$$

The PDF in equation 48 is valid for the whole interval below:

$$\frac{\sqrt{A_1^2+4A_0 a_2}}{2A_0}+\frac{A_1}{2A_0}-1 \le i \le \frac{\sqrt{A_1^2+4A_0 b_2}}{2A_0}+\frac{A_1}{2A_0}-1$$

3.3.2. Derivation of the PDF with the Transformation Method

$$f(A_2) = \frac{1}{b_2 - a_2} \qquad a_2 \le x \le b_2$$

$$f(4A_0A_2) = \frac{1}{4A_0}\frac{1}{b_2 - a_2} \qquad 4A_0a_2 \le x \le 4A_0b_2$$

$$f(A_1^2 + 4A_0A_2) = \frac{1}{4A_0}\frac{1}{b_2 - a_2} \qquad A_1^2 + \ 4A_0a_2 \le x \le A_1^2 + 4A_0b_2$$

$$let\ x = A_1^2 + \ 4A_0A_2\ and\ u = \sqrt{A_1^2 + \ 4A_0A_2}$$

$$h(u) = 2 * u * f(x) = 2u * \frac{1}{4A_0}\frac{1}{b_2 - a_2} \qquad \sqrt{A_1^2 + \ 4A_0a_2} \le u \le \sqrt{A_1^2 + \ 4A_0b_2}$$

$$h\left(\frac{u}{2A_0}\right) = 2\left(\frac{u}{\frac{1}{2A_0}}\right) * \left|\frac{1}{\frac{1}{2A_0}}\right| \frac{1}{4A_0}\frac{1}{b_2 - a_2}$$

$$h\left(\frac{u}{2A_0}\right) = 4A_0u * 2A_0\frac{1}{4A_0}\frac{1}{b_2 - a_2}$$

$$h\left(\frac{u}{2A_0}\right) = 2A_0u\frac{1}{b_2 - a_2} \qquad \frac{\sqrt{A_1^2 + \ 4A_0a_2}}{2A_0} \le u \le \frac{\sqrt{A_1^2 + \ 4A_0b_2}}{2A_0}$$

$$h\left(\frac{A_1}{2A_0} - 1 + \frac{u}{2A_0}\right) = 2A_0\left(u - \frac{A_1}{2A_0} + 1\right)\frac{1}{b_2 - a_2}$$

$$f(i) = \frac{2A_0}{b_2 - a_2}\left(i - \frac{A_1}{2A_0} + 1\right)$$

$$where\ \frac{A_1}{2A_0} - 1 + \frac{\sqrt{A_1^2 + \ 4A_0a_2}}{2A_0} \le i \le \frac{\sqrt{A_1^2 + \ 4A_0b_2}}{2A_0} + \frac{A_1}{2A_0} - 1$$

The results are found the same with the transformation of random variables method as in the distribution function method in equation 48 above.

3.3.3. The Expected Value

$$\int_{\frac{A_1}{2A_0}-1+\frac{\sqrt{A_1^2+4A_0a_2}}{2A_0}}^{\frac{\sqrt{A_1^2+4A_0b_2}}{2A_0}+\frac{A_1}{2A_0}-1} \frac{2A_0}{b_2-a_2}\left(i-\frac{A_1}{2A_0}+1\right) i\,di =$$

$$= \frac{2A_0}{b_2-a_2}\left(\frac{i^3}{3}-\frac{i^2}{2}\left(\frac{A_1}{2A_0}+1\right)\right)\Bigg|_{\frac{A_1}{2A_0}-1+\frac{\sqrt{A_1^2+4A_0a_2}}{2A_0}}^{\frac{\sqrt{A_1^2+4A_0b_2}}{2A_0}+\frac{A_1}{2A_0}-1}$$

$$= \frac{2A_0}{6(b_2-a_2)}\left(2i^3-3i^2\left(\frac{A_1}{2A_0}-1\right)\right)\Bigg|_{\frac{A_1}{2A_0}-1+\frac{\sqrt{A_1^2+4A_0a_2}}{2A_0}}^{\frac{\sqrt{A_1^2+4A_0b_2}}{2A_0}+\frac{A_1}{2A_0}-1}$$

$$= \frac{A_0}{3(b_2-a_2)}\frac{1}{2A_0}\left(4A_0i^3-3i^2(A_1-2A_0)\right)\Bigg|_{\frac{A_1}{2A_0}-1+\frac{\sqrt{A_1^2+4A_0a_2}}{2A_0}}^{\frac{\sqrt{A_1^2+4A_0b_2}}{2A_0}+\frac{A_1}{2A_0}-1}$$

$$= \frac{1}{6(b_2-a_2)}(4A_0i^3-3i^2A_1+6i^2A_0)\Bigg|_{\frac{A_1}{2A_0}-1+\frac{\sqrt{A_1^2+4A_0a_2}}{2A_0}}^{\frac{\sqrt{A_1^2+4A_0b_2}}{2A_0}+\frac{A_1}{2A_0}-1}$$

$$= \frac{i^2}{6(b_2-a_2)}(4A_0i-3A_1+6A_0)\Bigg|_{\frac{A_1}{2A_0}-1+\frac{\sqrt{A_1^2+4A_0a_2}}{2A_0}}^{\frac{\sqrt{A_1^2+4A_0b_2}}{2A_0}+\frac{A_1}{2A_0}-1}$$

$$= \frac{\left(\sqrt{A_1^2+\ 4A_0b_2}+A_1-2A_0\right)^2}{6(b_2-a_2)4{A_0}^2}\left(2A_1-4A_0+2\sqrt{A_1^2+\ 4A_0b_2}-3A_1+6A_0\right)$$

$$-\frac{\left(\sqrt{A_1^2+\ 4A_0a_2}+A_1-2A_0\right)^2}{6(b_2-a_2)4{A_0}^2}\left(2A_1-4A_0+2\sqrt{A_1^2+\ 4A_0a_2}-3A_1+6A_0\right)$$

$$= \frac{1}{6(b_2 - a_2)4{A_0}^2}\Bigg[\left(A_1^2 + 4A_0b_2 + 2(A_1 - 2A_0)\sqrt{A_1^2 + 4A_0b_2} + (A_1 - 2A_0)^2\right)\Big(-(A_1$$

$$- 2A_0) + 2\sqrt{A_1^2 + 4A_0b_2}\Big)$$

$$- \left(A_1^2 + 4A_0a_2 + 2(A_1 - 2A_0)\sqrt{A_1^2 + 4A_0a_2} + (A_1 - 2A_0)^2\right)\Big(-A_1 + 2A_0$$

$$+ 2\sqrt{A_1^2 + 4A_0a_2}\Big)\Bigg]$$

$$= \frac{1}{6(b_2 - a_2)4{A_0}^2}\Bigg[\Big(-(A_1 - 2A_0)(A_1^2 + 4A_0b_2) - 2(A_1 - 2A_0)^2\sqrt{A_1^2 + 4A_0b_2}$$

$$- (A_1 - 2A_0)^3 + 2(A_1^2 + 4A_0b_2)^{\frac{3}{2}} + 4(A_1 - 2A_0)(A_1^2 + 4A_0b_2)$$

$$+ 2(A_1 - 2A_0)^2\sqrt{A_1^2 + 4A_0b_2}\Big)$$

$$- \Big(-(A_1 - 2A_0)(A_1^2 + 4A_0a_2) - 2(A_1 - 2A_0)^2\sqrt{A_1^2 + 4A_0a_2} - (A_1 - 2A_0)^3$$

$$+ 2(A_1^2 + 4A_0a_2)^{\frac{3}{2}} + 4(A_1 - 2A_0)(A_1^2 + 4A_0a_2) + 2(A_1 - 2A_0)^2\sqrt{A_1^2 + 4A_0a_2}\Big)\Bigg]$$

$$= \frac{1}{6(b_2 - a_2)4{A_0}^2}\Bigg[\Big(3(A_1 - 2A_0)(A_1^2 + 4A_0b_2) - (A_1 - 2A_0)^3 + 2(A_1^2 + 4A_0b_2)^{\frac{3}{2}}\Big)$$

$$- \Big(3(A_1 - 2A_0)(A_1^2 + 4A_0a_2) - (A_1 - 2A_0)^3 + 2(A_1^2 + 4A_0a_2)^{\frac{3}{2}}\Big)\Bigg]$$

$$= \frac{1}{6(b_2 - a_2)4{A_0}^2}\Big[3(A_1 - 2A_0)(A_1^2 + 4A_0b_2) + 2(A_1^2 + 4A_0b_2)^{\frac{3}{2}}$$

$$- 3(A_1 - 2A_0)(A_1^2 + 4A_0a_2) - 2(A_1^2 + 4A_0a_2)^{\frac{3}{2}}\Big]$$

$$= \frac{1}{6(b_2 - a_2)4{A_0}^2}\Big[3(A_1 - 2A_0)(A_1^2 + 4A_0b_2 - A_1^2 - 4A_0a_2)$$

$$+ 2\left((A_1^2 + 4A_0b_2)^{\frac{3}{2}} - (A_1^2 + 4A_0a_2)^{\frac{3}{2}}\right)\Big]$$

$$= \frac{1}{6(b_2 - a_2)4{A_0}^2}\left[12A_0(A_1 - 2A_0)(b_2 - a_2) + 2\left((A_1^2 + 4A_0b_2)^{\frac{3}{2}} - (A_1^2 + 4A_0a_2)^{\frac{3}{2}}\right)\right]$$

$$= \frac{(A_1 - 2A_0)}{2A_0} + \frac{\left((A_1^2 + 4A_0b_2)^{\frac{3}{2}} - (A_1^2 + 4A_0a_2)^{\frac{3}{2}}\right)}{12(b_2 - a_2){A_0}^2}$$

$$E(i) = \frac{\boldsymbol{A_1}}{\boldsymbol{2A_0}} - \mathbf{1} + \frac{\left((\boldsymbol{A_1^2} + \boldsymbol{4A_0b_2})^{\frac{3}{2}} - (\boldsymbol{A_1^2} + \boldsymbol{4A_0a_2})^{\frac{3}{2}}\right)}{\mathbf{12}(\boldsymbol{b_2} - \boldsymbol{a_2}){\boldsymbol{A_0}}^2} \qquad (49)$$

3.3.4. The Variance

$$\sigma^2 = \int_{\frac{A_1}{2A_0}-1+\frac{\sqrt{A_1^2+4A_0a_2}}{2A_0}}^{\frac{\sqrt{A_1^2+4A_0b_2}}{2A_0}+\frac{A_1}{2A_0}-1} \frac{2A_0}{b_2 - a_2}\left(i - \frac{A_1}{2A_0} + 1\right) i^2\, di - \left(E(i)\right)^2 =$$

$$\sigma^2 = \frac{\boldsymbol{2A_0}}{\boldsymbol{b_2} - \boldsymbol{a_2}}\left(\frac{\boldsymbol{i^4}}{\mathbf{4}} - \frac{\boldsymbol{i^3}}{\mathbf{3}}\left(\frac{\boldsymbol{A_1}}{\boldsymbol{2A_0}} - \mathbf{1}\right)\right)\Bigg|_{\frac{A_1}{2A_0}-1+\frac{\sqrt{A_1^2+4A_0a_2}}{2A_0}}^{\frac{\sqrt{A_1^2+4A_0b_2}}{2A_0}+\frac{A_1}{2A_0}-1} - \left(\boldsymbol{E}(\boldsymbol{i})\right)^2 \qquad (50)$$

The variance of this case is left as an integral as a simplified formula could not be found. Less computational effort is required to find the expected value from the integral version.

3.4 Two Period IRR Problem: A_0 Constant, A_1 Constant, A_2 Exponential

A_0 and A_1 are deterministic values, and A_2 is exponentially distributed with a mean of μ where the rate of $\lambda = 1/\mu$.

$$f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

3.4.1. Derivation of the PDF with the Distribution Function Method

1. Step :

$$P(Z \leq z) = P\left(\sqrt{A + Bx} \leq z\right) = P(A + Bx \leq z^2) = P\left(x \leq \frac{z^2 - A}{B}\right)$$

$$= F_x\left(\frac{z^2 - A}{B}\right) \textit{ where } A\text{: } A_1^2 \text{ and B: } 4A_0 = \ F_x\left(\frac{z^2 - A_1^2}{4A_0}\right)$$

$$\textit{for exponential distribution } F_x = 1 - e^{-\lambda x}$$

$$= \ F_x\left(\frac{z^2 - A_1^2}{4A_0}\right) = 1 - e^{-\lambda(\frac{z^2 - A_1^2}{4A_0})}$$

$$F(Z) = 1 - e^{(\frac{-\lambda z^2 + \lambda A_1^2}{4A_0})}$$

$$\textit{where limits are}: 0 \leq \frac{z^2 - A_1^2}{4A_0} \leq \infty \text{, } \textit{thus } A_1 \ \leq z \leq \infty$$

2. Step

$$P\left(\frac{x}{A} \leq z\right) = P(x \leq zA) = F_x(2A_0 z)$$

$$F(Z) = 1 - e^{(\frac{-4\lambda A_0^2 z^2 + \lambda A_1^2}{4A_0})} = 1 - e^{(-\lambda A_0 z^2 + \frac{\lambda A_1^2}{4A_0})}$$

$$\text{The limits are } \ \frac{A_1}{2A_0} \ \leq z \leq \infty$$

3. Step

$$P(x + A \le i) = P(x \le i - A) = F_x\left(i - \frac{A_1}{2A_0} + 1\right)$$

$$F(i) = 1 - e^{\left(-\lambda A_0\left(i - \frac{A_1}{2A_0} + 1\right)^2 + \frac{\lambda A_1^2}{4A_0}\right)} \quad (51)$$

$$\text{The limits are } \frac{A_1}{A_0} - 1 \le i \le \infty$$

The CDF is checked for validity:

$$1 - e^{(-\lambda A_0\left(i - \frac{A_1}{2A_0} + 1\right)^2 + \frac{\lambda A_1^2}{4A_0})}\Bigg|_{\frac{A_1}{A_0}-1}^{\infty}$$

$$= 1 - e^{\left(-\lambda A_0\left(i - \frac{A_1}{2A_0} + 1\right)^2 + \frac{\lambda A_1^2}{4A_0}\right)}\Bigg|_{\frac{A_1}{A_0}-1}^{\infty}$$

$$= 1 - e^{\left(-\lambda A_0\left(\infty - \frac{A_1}{2A_0} + 1\right)^2 + \frac{\lambda A_1^2}{4A_0}\right)} - 1 + e^{\left(-\lambda A_0\left(\frac{A_1}{A_0} - 1 - \frac{A_1}{2A_0} + 1\right)^2 + \frac{\lambda A_1^2}{4A_0}\right)}$$

$$= e^{\left(-\lambda A_0\left(\frac{A_1}{2A_0}\right)^2 + \frac{\lambda A_1^2}{4A_0}\right)} - e^{(-\infty)}$$

$$= e^{\left(-\lambda A_0 \frac{A_1^2}{4A_0^2} + \frac{\lambda A_1^2}{4A_0}\right)} - e^{(-\infty)} = e^0 - e^{(-\infty)} = 1 - 0 = 1$$

The PDF is derived next:

$$f(i) = \frac{dF(i)}{di} = 2\lambda A_0\left(i - \frac{A_1}{2A_0} + 1\right)e^{(-\lambda A_0\left(i - \frac{A_1}{2A_0} + 1\right)^2 + \frac{\lambda A_1^2}{4A_0})} \quad (52)$$

The distribution function in equation 52 is valid for the interval:

$$\frac{A_1}{A_0} - 1 \le i \le \infty$$

3.4.2. Derivation of the PDF with the Transformation Method

$$f(A_2) = \lambda e^{-\lambda x} \qquad 0 \le x \le \infty$$

$$f(4A_0A_2) = \frac{1}{4A_0}\lambda e^{-\lambda\frac{u}{4A_0}} \qquad 0 \le u \le \infty$$

$$f(A_1^2 + 4A_0A_2) = \frac{1}{4A_0}\lambda e^{-\lambda\frac{u-A_1^2}{4A_0}} \qquad A_1^2 \le u \le \infty$$

$$let\ x = A_1^2 + 4A_0A_2\ and\ u = \sqrt{A_1^2 + 4A_0A_2}$$

$$h(u) = 2 * u * f(u^2) = 2u * \frac{1}{4A_0}\lambda e^{-\lambda\frac{u^2-A_1^2}{4A_0}} \qquad A_1 \le u \le \infty$$

$$h\left(\frac{u}{2A_0}\right) = 2\left(\frac{u}{\frac{1}{2A_0}}\right) * \left|\frac{1}{\frac{1}{2A_0}}\right| \frac{1}{4A_0}\lambda e^{-\lambda\frac{4{A_0}^2u^2-A_1^2}{4A_0}}$$

$$h\left(\frac{u}{2A_0}\right) = 4A_0u * 2A_0\frac{1}{4A_0}\lambda e^{-\lambda\frac{4{A_0}^2u^2-A_1^2}{4A_0}}$$

$$h\left(\frac{u}{2A_0}\right) = 2A_0u\lambda e^{-\lambda\frac{4{A_0}^2u^2-A_1^2}{4A_0}} \qquad \frac{A_1}{2A_0} \le u \le \infty$$

$$h\left(\frac{A_1}{2A_0} - 1 + \frac{u}{2A_0}\right) = 2A_0\left(u - \frac{A_1}{2A_0} + 1\right)\lambda e^{-\lambda\frac{4{A_0}^2\left(u-\frac{A_1}{2A_0}+1\right)^2-A_1^2}{4A_0}}$$

$$f(i) = 2\lambda A_0\left(i - \frac{A_1}{2A_0} + 1\right)e^{\left(-\lambda A_0\left(i-\frac{A_1}{2A_0}+1\right)^2+\frac{\lambda A_1^2}{4A_0}\right)}$$

$$where\ \frac{A_1}{A_0} - 1 \le i \le \infty$$

This solution for the PDF and the interval is the same as in equation 52.

3.4.3. The Expected Value

$$E(i) = \int_{\frac{A_1}{A_0}-1}^{\infty} 2\lambda A_0 \left(i - \frac{A_1}{2A_0} + 1\right) e^{\left(-\lambda A_0\left(i-\frac{A_1}{2A_0}+1\right)^2 + \frac{\lambda A_1^2}{4A_0}\right)} i di \tag{53}$$

3.4.4. The Variance

$$\sigma^2(i) = \int_{\frac{A_1}{A_0}-1}^{\infty} 2\lambda A_0 \left(i - \frac{A_1}{2A_0} + 1\right) e^{\left(-\lambda A_0\left(i-\frac{A_1}{2A_0}+1\right)^2 + \frac{\lambda A_1^2}{4A_0}\right)} i^2 di - [E(i)]^2 \tag{54}$$

The expected value and the variance of this case were attempted for simplification using the MATLAB. However, no simpler versions of these expressions could be obtained. Therefore, equations 53 and 54 are left as integrals. Equations 53 and 54 will result in numerical values if all applicable inputs are plugged in.

3.5 Two Period IRR Problem: A_0 Constant, A_1 Uniform, A_2 Constant

In this case A_0 and A_2 are assumed to be deterministic values and A_1 is uniformly distributed between [a_1, b_1] with the distribution function $f(A_1) = 1/(b_1 - a_1)$

3.5.1. Derivation of the PDF with the Distribution Function Method

$$i = \frac{-A_1 + 2A_0 - \sqrt{A_1{}^2 + 4A_0A_2}}{-2A_0}$$

$$i = \frac{A_1}{2A_0} + \frac{\sqrt{A_1{}^2 + 4A_0A_2}}{2A_0} - 1$$

$$i = \frac{A_1}{2A_0} + \sqrt{\frac{{A_1}^2}{4{A_0}^2} + \frac{4A_0A_2}{4{A_0}^2}} - 1$$

$$i = \frac{A_1}{2A_0} + \sqrt{\frac{{A_1}^2}{4{A_0}^2} + \frac{A_2}{A_0}} - 1$$

$$\text{Let } x = \frac{A_1}{2A_0} \quad \text{and} \quad c = \frac{A_2}{A_0}$$

Then,

$$i = x + \sqrt{x^2 + c} - 1$$

$$P\left(x + \sqrt{x^2 + c} - 1 \le i\right) = P\left(x + \sqrt{x^2 + c} \le i + 1\right) =$$

$$P\left(x - i - 1 \le -\sqrt{x^2 + c}\right) = P\left((x - i - 1)^2 \ge (-\sqrt{x^2 + c})^2\right) =$$

$$P(x^2 - 2x(i + 1) + (i + 1)^2 \ge x^2 + c) =$$

$$P(-2x(i + 1) + (i + 1)^2 \ge c) = P(-2x(i + 1) \ge c - (i + 1)^2) =$$

$$P\left(x \le \frac{c - (i + 1)^2}{-2(i + 1)}\right) = P\left(x \le \frac{(i + 1)^2 - c}{2(i + 1)}\right) = F(i)$$

$$f(x) = \frac{2A_0}{b_1 - a_1} \quad and\ F(x) = \frac{x - \frac{a_1}{2A_0}}{\frac{b_1}{2A_0} - \frac{a_1}{2A_0}}$$

$$\text{where the limits are } \frac{a_1}{2A_0} \le x \le \frac{b_1}{2A_0}$$

$$P\left(x \le \frac{(i+1)}{2} - \frac{c}{2(i+1)}\right) = P\left(x \le \frac{(i+1)}{2} - \frac{A_2}{2A_0(i+1)}\right) = F(i)$$

$$F(i) = \frac{\frac{(i+1)}{2} - \frac{A_2}{2A_0(i+1)} - \frac{a_1}{2A_0}}{\frac{b_1}{2A_0} - \frac{a_1}{2A_0}} = \frac{\frac{A_0(i+1)}{2A_0} - \frac{A_2}{2A_0(i+1)} - \frac{a_1}{2A_0}}{\frac{b_1}{2A_0} - \frac{a_1}{2A_0}} =$$

$$F(i) = \frac{\frac{A_0(i+1) - \frac{A_2}{(i+1)} - a_1}{2A_0}}{\frac{b_1 - a_1}{2A_0}} = \frac{\boldsymbol{A_0(i+1) - \frac{A_2}{(i+1)} - a_1}}{\boldsymbol{b_1 - a_1}} \qquad (55)$$

where the limits are:

$$\frac{a_1}{2A_0} \le \frac{(i+1)^2 - c}{2(i+1)} \le \frac{b_1}{2A_0}$$

$$\frac{a_1}{2A_0} = \frac{(i+1)^2 - c}{2(i+1)}$$

$$\frac{a_1}{A_0} = \frac{(i+1)^2 - c}{(i+1)}$$

$$a_1(i+1) = A_0(i+1)^2 - A_0 c$$

$$0 = A_0(i+1)^2 - a_1(i+1) - A_0 c$$

$$0 = A_0(i+1)^2 - a_1(i+1) - A_0 \frac{A_2}{A_0}$$

$$i + 1 = \frac{a_1 \pm \sqrt{{a_1}^2 + 4A_0A_2}}{2A_0}$$

$$i = \frac{a_1 \pm \sqrt{{a_1}^2 + 4A_0A_2}}{2A_0} - 1$$

The valid root is the positive root for i.

$$i_1 = \frac{a_1 + \sqrt{{a_1}^2 + 4A_0A_2}}{2A_0} - 1$$

The upper bound is calculated with the same steps above and it is,

$$i_2 = \frac{b_1 + \sqrt{{b_1}^2 + 4A_0A_2}}{2A_0} - 1$$

So, the limits are:

$$\frac{a_1 + \sqrt{{a_1}^2 + 4A_0A_2}}{2A_0} - 1 \leq i \leq \frac{b_1 + \sqrt{{b_1}^2 + 4A_0A_2}}{2A_0} - 1$$

CDF is checked for verification:

$$\left. \frac{A_0(i+1) - \dfrac{A_2}{(i+1)} - a_1}{b_1 - a_1} \right|_{\frac{a_1+\sqrt{{a_1}^2+4A_0A_2}}{2A_0}-1}^{\frac{b_1+\sqrt{{b_1}^2+4A_0A_2}}{2A_0}-1} = 1$$

This expression could not be simplified to 1 as the square roots still exist in the calculations. However when applicable numerical values are plugged in MATLAB, it could be seen that the CDF equals 1.

The PDF is derived next:

$$f(i) = \frac{d}{di}F(i) = \frac{d}{di}\left[\frac{A_0(i+1) - \frac{A_2}{(i+1)} - a_1}{b_1 - a_1}\right]$$

$$f(i) = \frac{1}{b_1 - a_1}\left[A_0 + \frac{A_2}{(i+1)^2}\right] \tag{56}$$

3.5.2. The Expected Value

$$\int_{\frac{a_1+\sqrt{a_1{}^2-4A_0A_2}}{2A_0}-1}^{\frac{b_1+\sqrt{b_1{}^2-4A_0A_2}}{2A_0}-1} \frac{1}{b_1 - a_1}\left[A_0 + \frac{A_2}{(1+i)^2}\right] i\, di =$$

$$E(i) = \frac{\mathbf{1}}{\boldsymbol{b_1 - a_1}}\left[\boldsymbol{A_0}\frac{\boldsymbol{i^2}}{\mathbf{2}} + \frac{\boldsymbol{A_2}}{\boldsymbol{(1+i)}} + \boldsymbol{A_2 ln\,(1+i)}\right]\Bigg|_{\frac{a_1+\sqrt{a_1{}^2-4A_0A_2}}{2A_0}-1}^{\frac{b_1+\sqrt{b_1{}^2-4A_0A_2}}{2A_0}-1} \tag{57}$$

3.5.3. The Variance

$$\int_{\frac{a_1+\sqrt{a_1{}^2-4A_0A_2}}{2A_0}-1}^{\frac{b_1+\sqrt{b_1{}^2-4A_0A_2}}{2A_0}-1} \frac{1}{b_1 - a_1}\left[A_0 + \frac{A_2}{(1+i)^2}\right] i^2 di - [E(i)]^2 =$$

$$\sigma^2(i) = \frac{\mathbf{1}}{\boldsymbol{b_1 - a_1}}\left[\boldsymbol{A_0}\frac{\boldsymbol{i^3}}{\mathbf{3}} + \boldsymbol{A_2}\left(\frac{\boldsymbol{i(i+2)}}{\boldsymbol{(1+i)}} - \boldsymbol{2 ln(1+i)}\right)\right]\Bigg|_{\frac{a_1+\sqrt{a_1{}^2-4A_0A_2}}{2A_0}-1}^{\frac{b_1+\sqrt{b_1{}^2-4A_0A_2}}{2A_0}-1} - [\boldsymbol{E(i)}]^2 \tag{58}$$

Again, the expected value and the variance expressions could not be simplified; therefore they are left as integrals.

3.6 Two Period IRR Problem: A_0 Constant, A_1 Exponential, A_2 Constant

In this case A_0 and A_2 are assumed to be deterministic values and A_1 is exponentially distributed with rate λ with the distribution function

$$f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

3.6.1. Derivation of the PDF with the Distribution Function Method

$$i = \frac{A_1}{2A_0} + \sqrt{\frac{{A_1}^2}{4{A_0}^2} + \frac{A_2}{A_0}} - 1$$

$$\text{Let } x = \frac{A_1}{2A_0} \quad \text{and} \quad c = \frac{A_2}{A_0}$$

$$F_x = 1 - e^{-2A_0\lambda x} \qquad 0 \leq x \leq \infty$$

Then

$$i = x + \sqrt{x^2 + c} - 1$$

$$P\left(x + \sqrt{x^2 + c} - 1 \leq i\right) = P\left(x \leq \frac{(i+1)^2 - c}{2(i+1)}\right)$$

$$P\left(x \leq \frac{(i+1)}{2} - \frac{c}{2(i+1)}\right) = P\left(x \leq \frac{(i+1)}{2} - \frac{A_2}{2A_0(i+1)}\right) = F(i)$$

$$F(i) = 1 - e^{-2A_0\lambda i} = 1 - e^{-2A_0\lambda\left(\frac{(i+1)}{2} - \frac{A_2}{2A_0(i+1)}\right)} = 1 - e^{\left(-A_0\lambda(i+1) + \frac{\lambda A_2}{(i+1)}\right)}$$

$$F(i) = 1 - e^{\left(-A_0\lambda(i+1) + \frac{\lambda A_2}{(i+1)}\right)} \qquad (59)$$

Where the limits are:

$$0 \leq \frac{(i+1)^2 - c}{2(i+1)} \leq \infty$$

$$0 = \frac{(i+1)^2 - c}{2(i+1)}$$

$$c = (i+1)^2$$

$$\frac{A_2}{A_0} = (i+1)^2$$

$$\sqrt{\frac{A_2}{A_0}} - 1 = i_1$$

$$\infty = \frac{(i+1)^2 - c}{2(i+1)}$$

$$i_2 = \infty$$

So the limits are:

$$\sqrt{\frac{A_2}{A_0}} - 1 \leq i \leq \infty$$

The CDF is checked for verification:

$$1 - e^{\left(-A_0\lambda(i+1) + \frac{\lambda A_2}{(i+1)}\right)}\Bigg|_{\sqrt{\frac{A_2}{A_0}}-1}^{\infty} =$$

$$= 1 - e^{\left(-\infty + \frac{\lambda A_2}{\infty}\right)} - 1 + e^{\left(-A_0\lambda\left(\sqrt{\frac{A_2}{A_0}}-1+1\right)+\frac{\lambda A_2}{\left(\sqrt{\frac{A_2}{A_0}}-1+1\right)}\right)}$$

$$= e^{\left(-\lambda\left(\sqrt{A_0A_2}\right)+\lambda\sqrt{A_0A_2}\right)} - e^{(-\infty)} = e^0 - e^{-\infty} = 1 - 0 = 1$$

The PDF is derived next:

$$f(i) = \frac{d}{di}F(i) = \frac{d}{di}\left[1 - e^{\left(-A_0\lambda(i+1)+\frac{\lambda A_2}{(i+1)}\right)}\right]$$

$$f(i) = \left(A_0\lambda + \frac{\lambda A_2}{(i+1)^2}\right)e^{\left(-A_0\lambda(i+1)+\frac{\lambda A_2}{(i+1)}\right)} \tag{60}$$

3.6.2. The Expected Value

$$E(i) = \int_{\sqrt{\frac{A_2}{A_0}}-1}^{\infty} \left(A_0\lambda + \frac{\lambda A_2}{(i+1)^2}\right)e^{\left(-A_0\lambda(i+1)+\frac{\lambda A_2}{(i+1)}\right)} i\, di$$

3.6.3. The Variance

$$\sigma^2 = \int_{\sqrt{\frac{A_2}{A_0}}-1}^{\infty} \left(A_0\lambda + \frac{\lambda A_2}{(i+1)^2}\right)e^{\left(-A_0\lambda(i+1)+\frac{\lambda A_2}{(i+1)}\right)} i^2\, di - [E(i)]^2$$

Numerical values can be obtained in MATLAB for expectation and variance.

Chapter 4 – Results

The analytical solutions for the PDF, the CDF, the expected value and the variance are displayed in Table 9, Table 10, and Table 11 respectively. In this chapter, the analytical functions are calculated for numerical examples for each case. The solutions found by @RISK are also used in order to validate analytical results. The graphs of the functions provided clearly display the same pattern for each case.

Table 9 . Summary of the PDF of IRR for All Cases

	CASES	IRR	PDF	LIMITS
ONE PERIOD PROBLEM	A_0 constant, A_1 uniform	$i=\frac{A_1}{A_0}-1$	$\frac{A_0}{b_1-a_1}$	$\frac{a_1}{A_0}-1\le i\le\frac{b_1}{A_0}-1$
	A_0 uniform, A_1 uniform	$i=\frac{A_1}{A_0}-1$	$\frac{1}{2}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}({b_0}^2-\frac{{a_1}^2}{(i+1)^2})$	$\frac{a_1}{b_0}-1\le i<\frac{a_1}{a_0}-1$
			$\frac{1}{2}(\frac{b_0+a_0}{b_1-a_1})$	$\frac{a_1}{a_0}-1\le i<\frac{b_1}{b_0}-1$
			$-\frac{1}{2}\frac{1}{b_0-a_0}\frac{1}{b_1-a_1}\left({a_0}^2-\frac{{b_1}^2}{(i+1)^2}\right)$	$\frac{b_1}{b_0}-1\le i\le\frac{b_1}{a_0}-1$
TWO PERIOD PROBLEM	A_0 constant, A_1 constant, A_2 uniform	$i=\frac{-A_1+2A_0-\sqrt{{A_1}^2+4A_0A_2}}{-2A_0}$	$\frac{2A_0}{b_2-a_2}\left(i-\frac{A_1}{2A_0}+1\right)$	$\frac{\sqrt{\mathrm{A}_1^2+4\mathrm{A}_0a_2}}{2\mathrm{A}_0}+\frac{A_1}{2\mathrm{A}_0}-1\le z\le\frac{\sqrt{\mathrm{A}_1^2+4\mathrm{A}_0b_2}}{2\mathrm{A}_0}+\frac{A_1}{2\mathrm{A}_0}-1$
	A_0 constant, A_1 constant, A_2 exponential	$i=\frac{-A_1+2A_0-\sqrt{{A_1}^2+4A_0A_2}}{-2A_0}$	$2\lambda A_0\left(i-\frac{A_1}{2A_0}+1\right)e^{\left(-\lambda A_0\left(i-\frac{A_1}{2A_0}+1\right)^2+\frac{\lambda A_1^2}{4A_0}\right)}$	$\frac{A_1}{A_0}-1\le i\le\infty$
	A_0 constant, A_1 uniform, A_2 constant	$i=\frac{-A_1+2A_0-\sqrt{{A_1}^2+4A_0A_2}}{-2A_0}$	$\frac{1}{b_1-a_1}\left[A_0+\frac{A_2}{(i+1)^2}\right]$	$\frac{a_1+\sqrt{{a_1}^2-4A_0A_2}}{2A_0}-1\le i\le\frac{b_1+\sqrt{{b_1}^2-4A_0A_2}}{2A_0}-1$
	A_0 constant, A_1 exponential, A_2 constant	$i=\frac{-A_1+2A_0-\sqrt{{A_1}^2+4A_0A_2}}{-2A_0}$	$\left(A_0\lambda+\frac{\lambda A_2}{(i+1)^2}\right)e^{\left(-A_0\lambda(i+1)+\frac{\lambda A_2}{(i+1)}\right)}$	$\sqrt{\frac{A_2}{A_0}}-1\le i\le\infty$

Table 10 . Summary of the CDF of IRR for All Cases

	CASES	IRR	CDF	LIMITS
ONE PERIOD PROBLEM	A_0 constant, A_1 uniform	$i = \frac{A_1}{A_0} - 1$	$\frac{(i+1)A_0 - a_1}{b_1 - a_1}$	$\frac{a_1}{A_0} - 1 \leq i \leq \frac{b_1}{A_0} - 1$
	A_0 uniform, A_1 uniform	$i = \frac{A_1}{A_0} - 1$	$\frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}(b_0{}^2(i+1) - 2b_0a_1 + \frac{a_1{}^2}{(i+1)})$	$\frac{a_1}{b_0} - 1 \leq i < \frac{a_1}{a_0} - 1$
			$\frac{1}{2}\frac{1}{b_1 - a_1}[(b_0 + a_0)(i+1) - 2a_1]$	$\frac{a_1}{a_0} - 1 \leq i < \frac{b_1}{b_0} - 1$
			$1 - \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{b_1{}^2}{(i+1)} - 2b_1a_0 + a_0{}^2(i+1)\right)$	$\frac{b_1}{b_0} - 1 \leq i \leq \frac{b_1}{a_0} - 1$
TWO PERIOD PROBLEM	A_0 constant, A_1 constant, A_2 uniform	$i = \frac{-A_1 + 2A_0 - \sqrt{A_1{}^2 + 4A_0A_2}}{-2A_0}$	$\frac{4A_0{}^2(i - \frac{A_1}{2A_0} + 1)^2 - A_1^2 - 4A_0a_2}{4A_0(b_2 - a_2)}$	$\frac{\sqrt{A_1^2 + 4A_0a_2}}{2A_0} + \frac{A_1}{2A_0} - 1 \leq z \leq \frac{\sqrt{A_1^2 + 4A_0b_2}}{2A_0} + \frac{A_1}{2A_0} - 1$
	A_0 constant, A_1 constant, A_2 exponential	$i = \frac{-A_1 + 2A_0 - \sqrt{A_1{}^2 + 4A_0A_2}}{-2A_0}$	$1 - e^{(-\lambda A_0\left(z - \frac{A_1}{2A_0} + 1\right)^2 + \frac{\lambda A_1^2}{4A_0})}$	$\frac{A_1}{A_0} - 1 \leq i \leq \infty$
	A_0 constant, A_1 uniform, A_2 constant	$i = \frac{-A_1 + 2A_0 - \sqrt{A_1{}^2 + 4A_0A_2}}{-2A_0}$	$\frac{A_0(i+1) - \frac{A_2}{(i+1)} - a_1}{b_1 - a_1}$	$\frac{a_1 + \sqrt{a_1{}^2 - 4A_0A_2}}{2A_0} - 1 \leq i \leq \frac{b_1 + \sqrt{b_1{}^2 - 4A_0A_2}}{2A_0} - 1$
	A_0 constant, A_1 exponential, A_2 constant	$i = \frac{-A_1 + 2A_0 - \sqrt{A_1{}^2 + 4A_0A_2}}{-2A_0}$	$1 - e^{\left(-A_0\lambda(i+1) + \frac{\lambda A_2}{(i+1)}\right)}$	$\sqrt{\frac{A_2}{A_0}} - 1 \leq i \leq \infty$

Table 11 . Summary of the Expected Values and Variances of IRR

	CASES	EXPECTED VALUE	VARIANCE
ONE PERIOD	A_0 constant, A_1 uniform	$\frac{b_1+a_1}{2A_0}-1$	$\frac{1}{12{A_0}^2}(b_1-a_1)^2$
	A_0 uniform, A_1 uniform	$E(i)=-1-\frac{1}{2}\frac{(a_1+b_1)}{b_0-a_0}ln\left(\frac{a_0}{b_0}\right)$	$\frac{\left({b_1}^2+a_1b_1+{a_1}^2\right)}{3a_0b_0}-\frac{1}{4}\left(\frac{(a_1+b_1)}{b_0-a_0}\right)^2 ln^2\left(\frac{a_0}{b_0}\right)$
TWO PERIOD PROBLEM	A_0 constant, A_1 constant, A_2 uniform	$\frac{A_1}{2A_0}-1+\frac{\left((A_1^2+4A_0b_2)^{\frac{3}{2}}-(A_1^2+4A_0a_2)^{\frac{3}{2}}\right)}{12(b_2-a_2){A_0}^2}$	$\frac{2A_0}{b_2-a_2}\left(\frac{i^4}{4}-\frac{i^3}{3}\left(\frac{A_1}{2A_0}+1\right)\right)\Bigg\|_{\frac{A_1}{2A_0}-1+\frac{\sqrt{A_1^2+4A_0a_2}}{2A_0}}^{\frac{\sqrt{A_1^2+4A_0b_2}}{2A_0}+\frac{A_1}{2A_0}-1}-\left(E(i)\right)^2$
	A_0 constant, A_1 constant, A_2 exponential	$\int_{\frac{A_1}{A_0}-1}^{\infty}2\lambda A_0\left(i-\frac{A_1}{2A_0}+1\right)e^{\left(-\lambda A_0\left(i-\frac{A_1}{2A_0}+1\right)^2+\frac{\lambda A_1^2}{4A_0}\right)}idi$	$\int_{\frac{A_1}{A_0}-1}^{\infty}2\lambda A_0\left(i-\frac{A_1}{2A_0}+1\right)e^{\left(-\lambda A_0\left(i-\frac{A_1}{2A_0}+1\right)^2+\frac{\lambda A_1^2}{4A_0}\right)}i^2di-[E(i)]^2$
	A_0 constant, A_1 uniform, A_2 constant	$\frac{1}{b_1-a_1}\left[A_0\frac{i^2}{2}+\frac{A_2}{(1+i)}+A_2ln\ (1+i)\right]\Bigg\|_{\frac{a_1+\sqrt{{a_1}^2-4A_0A_2}}{2A_0}-1}^{\frac{b_1+\sqrt{{b_1}^2-4A_0A_2}}{2A_0}-1}$	$\frac{1}{b_1-a_1}\left[A_0\frac{i^3}{3}+A_2\left(\frac{i(i+2)}{(1+i)}-2ln(1+i)\right)\right]\Bigg\|_{\frac{a_1+\sqrt{{a_1}^2-4A_0A_2}}{2A_0}-1}^{\frac{b_1+\sqrt{{b_1}^2-4A_0A_2}}{2A_0}-1}-[E(i)]^2$
	A_0 constant, A_1 exponential, A_2 constant	$\int_{\sqrt{\frac{A_2}{A_0}}-1}^{\infty}\left(A_0\lambda+\frac{\lambda A_2}{(i+1)^2}\right)e^{\left(-A_0\lambda(i+1)+\frac{\lambda A_2}{(i+1)}\right)}i\ di$	$\int_{\sqrt{\frac{A_2}{A_0}}-1}^{\infty}\left(A_0\lambda+\frac{\lambda A_2}{(i+1)^2}\right)e^{\left(-A_0\lambda(i+1)+\frac{\lambda A_2}{(i+1)}\right)}i^2\ di-[E(i)]^2$

4.1 Numerical Example for One Period IRR Problem: A_0 Constant, A_1 Uniform

Let $A_0 = 120\ and\ A_1 \sim \text{uniform}(a_1 = 125, b_1 = 175)$

The PDF of this case is:

$$f(i) = \frac{A_0}{b_1 - a_1}$$

Then

$$f(i) = \frac{120}{175 - 125} = 2.4$$

This value means that the function shows a constant value of 2.4 in the entire period.

The CDF is:

$$F(i) = \frac{(i+1)A_0 - a_1}{b_1 - a_1} = \frac{120(i+1) - 125}{175 - 125} = \frac{120i - 5}{50}$$

Where the limits are:

$$\frac{a_1}{A_0} - 1 \leq i \leq \frac{b_1}{A_0} - 1$$

$$\frac{125}{120} - 1 \leq i \leq \frac{175}{120} - 1$$

$$0.04167 \leq i \leq 0.4583$$

Expected value is:

$$E(i) = \frac{b_1 + a_1}{2A_0} - 1 = \frac{175 + 125}{2 * 120} - 1 = 0.25$$

Variance is:

$$\sigma^2 = \frac{(b_1 - a_1)^2}{12{A_0}^2} = \frac{(175 - 125)^2}{12 * 120^2} = 0.014468$$

The PDF and the CDF are graphed in MATLAB in Figure 8 and Figure 9 below.

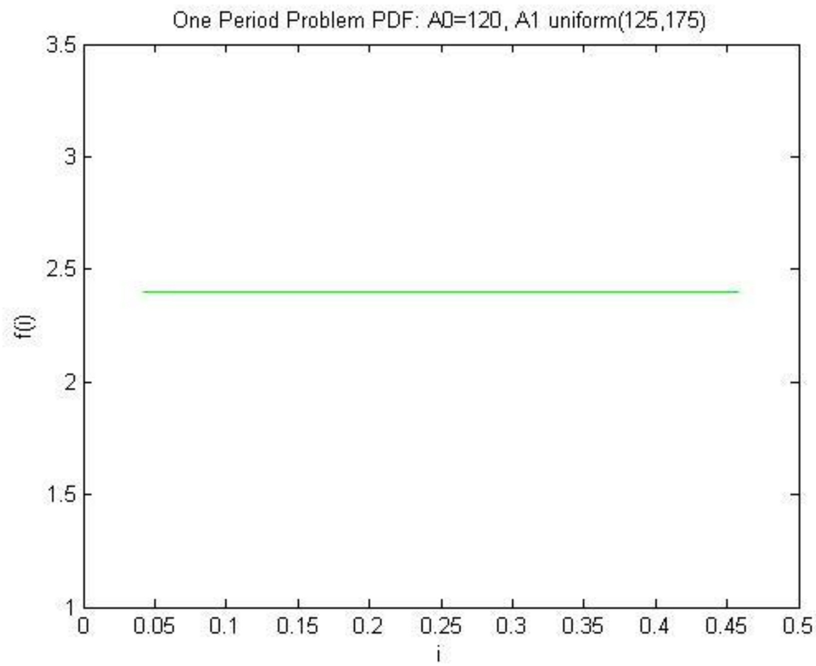

Figure 8 . One Period Problem PDF Graph: A0=120, A1 Uniform (125, 175)

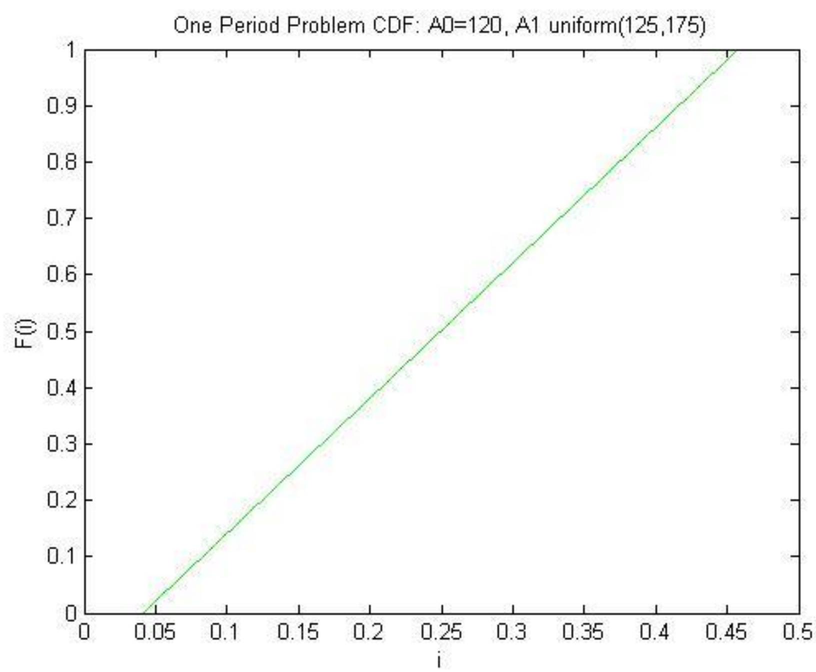

Figure 9 . One Period Problem CDF Graph: A0=120, A1 Uniform (125, 175)

The graphs found using the @RISK simulation software verify that the analytical graphs are correct as shown in Figure 10 and Figure 11.

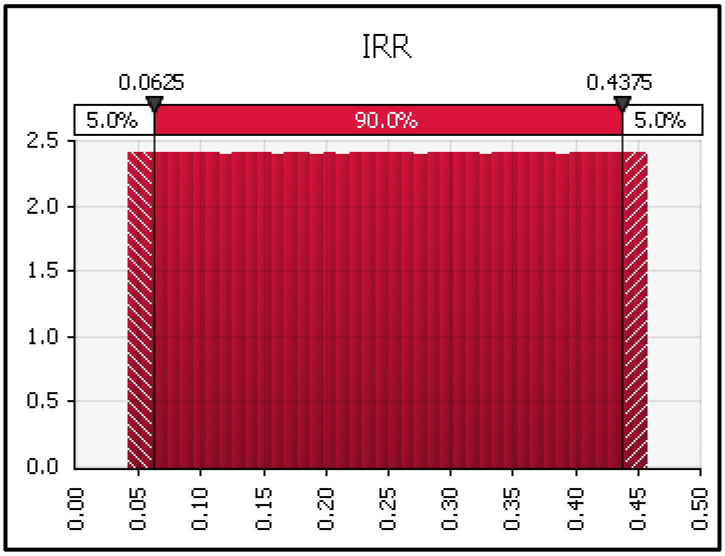

Figure 10 . One Period Problem Simulation of PDF: A0=120, A1 Uniform (125, 175)

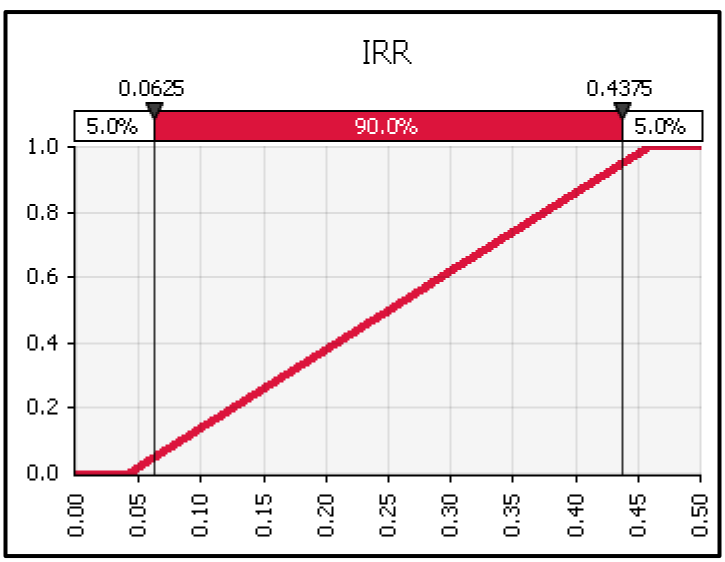

Figure 11 . One Period Problem Simulation of CDF: A0=120, A1 Uniform (125, 175)

The solutions of @RISK show that the analytical results verify the Monte Carlo Simulation. The expected value and the variance of the simulation are the same as the ones found analytically as shown in Table 12. In all simulations, 10000 iterations were use.

Table 12 . One Period Problem Simulation Statistics: A0=120, A1 Uniform (125,175)

Summary Statistics for IRR	
Minimum	0.04171
Maximum	0.45832
Mean	0.25000
Std Dev	0.12029
Variance	0.01446902

Simulation results, shown above also include the extreme values, which are really calculated analytically as the upper and lower limits of the function.

4.2 Numerical Example for One Period IRR Problem: A_0 Uniform, A_1 Uniform

In this case, the uniform distributions for the two random variables are assumed not to be overlapping. Let $A_0 \sim \text{uniform}(a_0 = 80, b_0 = 100)$ and $A_1 \sim \text{uniform}(a_1 = 120, b_1 = 200)$

Interval 1 : $\frac{a_1}{b_0} - 1 \leq i < \frac{a_1}{a_0} - 1$

$$\frac{120}{100} - 1 \leq i < \frac{120}{80} - 1$$

$$0.2 \leq i < 0.5$$

$$f_1(i) = \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}({b_0}^2 - \frac{{a_1}^2}{(i+1)^2})$$

$$f_1(i) = \frac{1}{2}\frac{1}{100 - 80}\frac{1}{200 - 120}(100^2 - \frac{120^2}{(i+1)^2})$$

$$f_1(i) = \frac{1}{3200}(100^2 - \frac{120^2}{(i+1)^2})$$

$$f_1(i) = \left(\frac{25}{8} - \frac{9}{2(i+1)^2}\right)$$

Interval 2 : $\frac{a_1}{a_0} - 1 \leq i < \frac{b_1}{b_0} - 1$

$$\frac{120}{80} - 1 \leq i < \frac{200}{100} - 1$$

$$0.5 \leq i < 1$$

$$f_2(i) = \frac{1}{2}\left(\frac{b_0 + a_0}{b_1 - a_1}\right) = \frac{1}{2}\left(\frac{100 + 80}{200 - 120}\right) = 1.125$$

In this interval, the PDF of the IRR has a constant value as it can be seen in the Figure 12.

Interval 3 : $\frac{b_1}{b_0} - 1 \leq i \leq \frac{b_1}{a_0} - 1$

$$\frac{200}{100} - 1 \leq i \leq \frac{200}{80} - 1$$

$$1 \leq i \leq 1.5$$

$$f_3(i) = -\frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left({a_0}^2 - \frac{{b_1}^2}{(i+1)^2}\right)$$

$$f_3(i) = -\frac{1}{2}\frac{1}{100 - 80}\frac{1}{200 - 120}\left(80^2 - \frac{200^2}{(i+1)^2}\right)$$

$$f_3(i) = \left(\frac{25}{2(i+1)^2} - 2\right)$$

The PDF of IRR is drawn using MATLAB as shown in Figure 12. The intervals can also be observed from the graph. Simulation graph shows a very similar pattern to the PDF found analytically it can be seen in Figure 13.

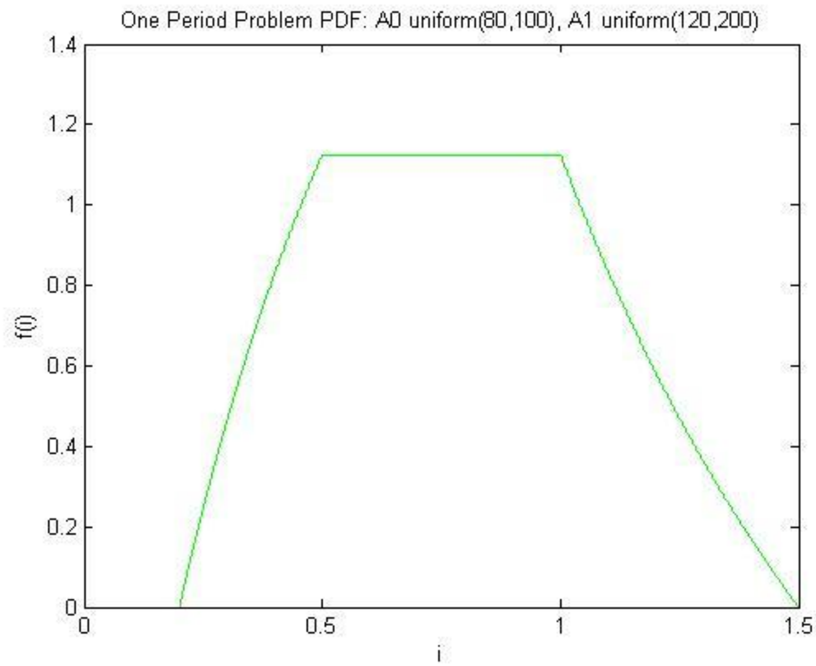

Figure 12 . One Period Problem PDF Graph: A0 Uniform (80, 100), A1 Uniform (120, 200)

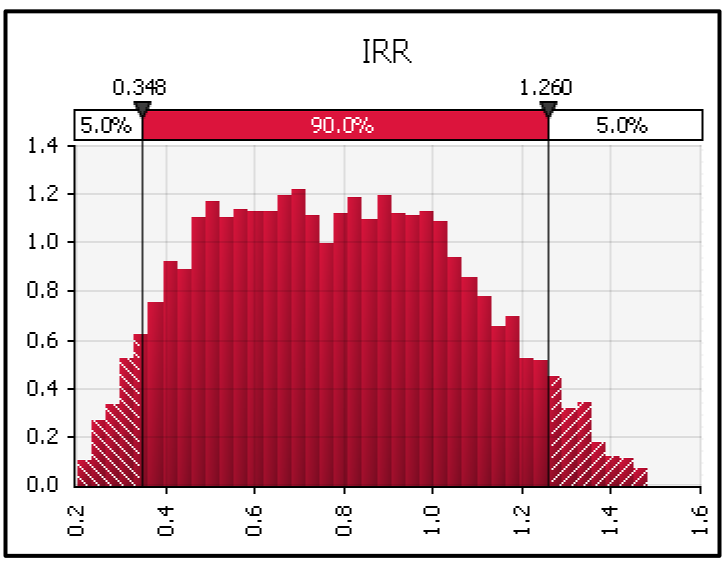

Figure 13 . One Period Problem Simulation of PDF: A0 Uniform (80, 100), A1 Uniform (120, 200)

The CDF has the functions in each interval as follows:

Interval 1 : $0.2 \leq i < 0.5$

$$F_1(i) = \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}({b_0}^2(i+1) - 2b_0 a_1 + \frac{{a_1}^2}{(i+1)})$$

$$F_1(i) = \frac{1}{2}\frac{1}{100 - 80}\frac{1}{200 - 120}\left(100^2(i+1) - 2 * 100 * 120 + \frac{120^2}{(i+1)}\right)$$

$$F_1(i) = \left(\frac{100^2(i+1)}{3200} - \frac{2 * 100 * 120}{3200} + \frac{\frac{120^2}{(i+1)}}{3200}\right)$$

$$F_1(i) = \left(\frac{25(i+1)}{8} - \frac{30}{4} + \frac{9}{2(i+1)}\right)$$

Interval 2 : $0.5 \leq i < 1$

$$F_2(i) = \frac{1}{2}\frac{1}{b_1 - a_1}[(b_0 + a_0)(i+1) - 2a_1]$$

$$F_2(i) = \frac{1}{2}\frac{1}{200 - 120}[(100 + 80)(i+1) - 2 * 120]$$

$$F_2(i) = \frac{1}{160}[180(i+1) - 240]$$

$$F_2(i) = \left[\frac{180(i+1)}{160} - \frac{240}{160}\right]$$

$$F_2(i) = \left[\frac{9(i+1)}{8} - \frac{3}{2}\right]$$

Interval 3 : $1 \leq i \leq 1.5$

$$F_3(i) = 1 - \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{{b_1}^2}{(i+1)} - 2b_1a_0 + {a_0}^2(i+1)\right)$$

$$F_3(i) = 1 - \frac{1}{2}\frac{1}{100 - 80}\frac{1}{200 - 120}\left(\frac{200^2}{(i+1)} - 2 * 200 * 80 + 80^2(i+1)\right)$$

$$F_3(i) = 1 - \left(\frac{200^2}{3200(i+1)} - \frac{2 * 200 * 80}{3200} + \frac{80^2(i+1)}{3200}\right)$$

$$F_3(i) = 1 - \left(\frac{25}{4(i+1)} - 10 + 2(i+1)\right)$$

The CDF graph obtained for this example is given in Figure 14. The simulation result plot in Figure 15 is also given to check that the solution is correct.

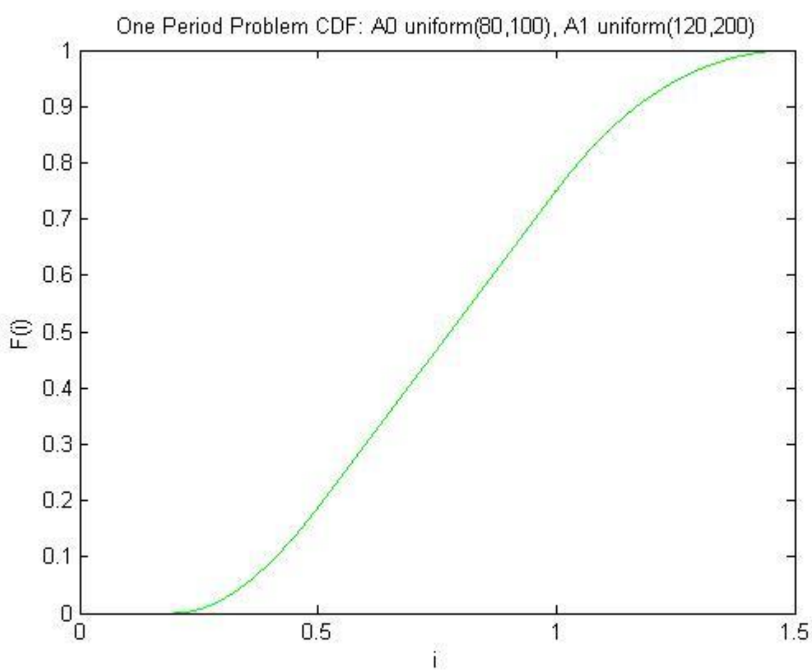

Figure 14 . One Period Problem CDF Graph: A0 Uniform (80, 100), A1 Uniform (120, 200)

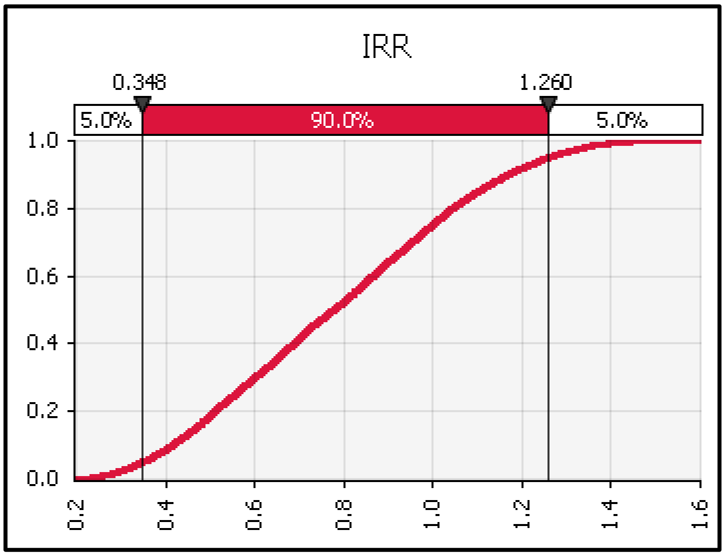

Figure 15 . One Period Problem Simulation of CDF: A0 Uniform (80, 100), A1 Uniform (120, 200)

Expectation:

$$E(i) = -1 - \frac{1}{2}\frac{(a_1 + b_1)}{b_0 - a_0} ln\left(\frac{a_0}{b_0}\right)$$

$$E(i) = -1 - \frac{1}{2}\frac{(120 + 200)}{100 - 80} ln\left(\frac{80}{100}\right)$$

$$E(i) = -1 - 8ln\left(\frac{80}{100}\right) = 0.785$$

Variance:

$$\sigma^2(i) = \frac{\left({b_1}^2 + a_1 b_1 + {a_1}^2\right)}{3a_0 b_0} - \frac{1}{4}\left(\frac{(a_1 + b_1)}{b_0 - a_0}\right)^2 ln^2\left(\frac{a_0}{b_0}\right)$$

$$\sigma^2(i) = \frac{(200^2 + 120 * 200 + 120^2)}{3 * 80 * 100} - \frac{1}{4}\left(\frac{(120 + 200)}{100 - 80}\right)^2 ln^2\left(\frac{80}{100}\right)$$

$$\sigma^2(i) = \frac{(400 + 240 + 144)}{3 * 80} - 64ln^2(0.8)$$

$$\sigma^2(i) = \frac{784}{240} - 64ln^2(0.8) = 0.079$$

These computed results are also found by the simulation. The results of the simulation can be seen in Table 13.

Table 13 . One Period Problem Simulation Statistics: A0 Uniform (80, 100) A1 Uniform (120, 200)

Summary Statistics for IRR	
Minimum	0.20336
Maximum	1.48429
Mean	0.78492
Std Dev	0.28133
Variance	0.079147875

4.3 Numerical Example for Two Period IRR Problem: A_0 Constant, A_1 Constant, A_2 Uniform

Let $A_0 = 200,\ A_1 = 120\ and\ A_2 \sim \text{uniform}(a_2 = 125, b_2 = 175)$

$$\frac{\sqrt{A_1^2 + 4A_0 a_2}}{2A_0} + \frac{A_1}{2A_0} - 1 \le i \le \frac{\sqrt{A_1^2 + 4A_0 b_2}}{2A_0} + \frac{A_1}{2A_0} - 1$$

$$\frac{\sqrt{120^2 + 4 * 200 * 125}}{2 * 200} + \frac{120}{2 * 200} - 1 \le i \le \frac{\sqrt{120^2 + 4 * 200 * 175}}{2 * 200} + \frac{120}{2 * 200} - 1$$

$$0.1455 \le i \le 0.2823$$

$$f(i) = \frac{2A_0}{b_2 - a_2}(i - \frac{A_1}{2A_0} + 1)$$

$$f(i) = \frac{2 * 200}{175 - 125}(i - \frac{120}{2 * 200} + 1)$$

$$f(i) = 8(i + 0.7)$$

The graph of the PDF using MATLAB and @RISK are shown in Figure 16 and Figure 17.

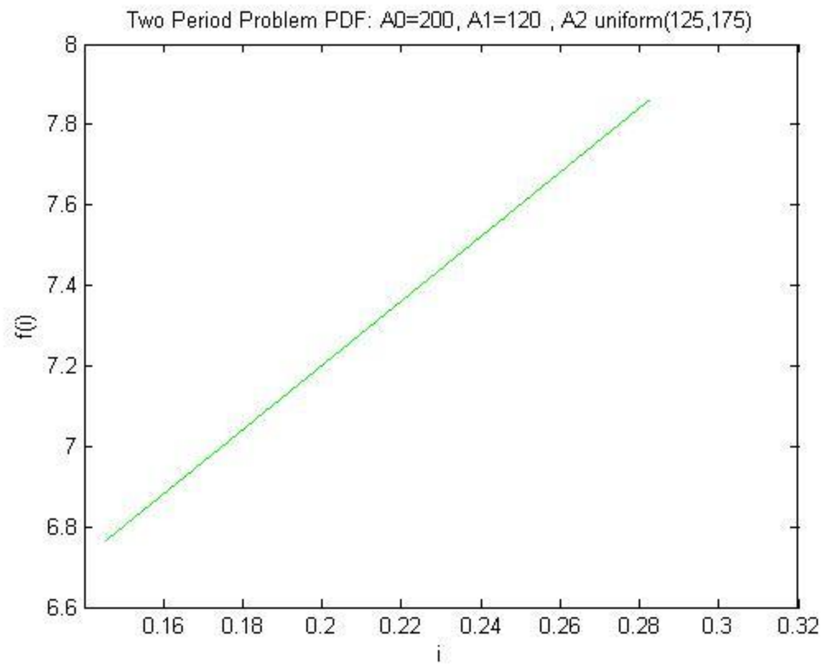

Figure 16 . Two Period Problem PDF Graph: A0=200, A1=120, A2 Uniform (125, 175)

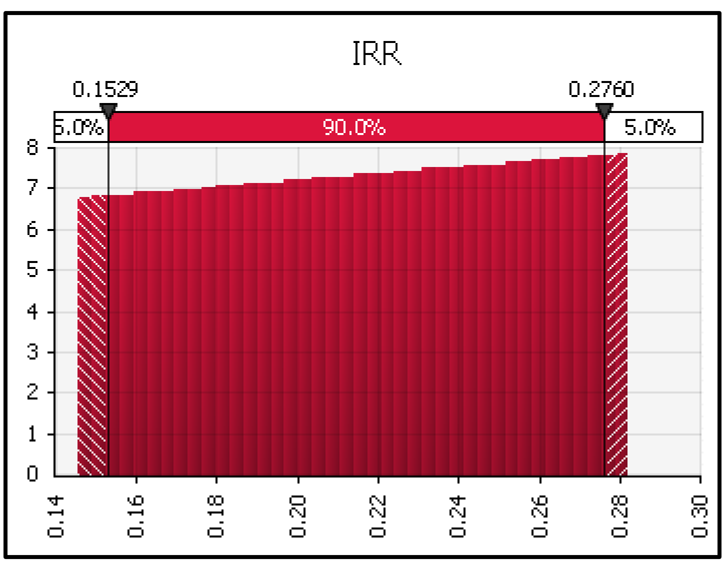

Figure 17 . Two Period Problem Simulation of PDF: A0=200, A1=120, A2 Uniform (125, 175)

$$F(i) = \frac{4{A_0}^2(i - \frac{A_1}{2A_0} + 1)^2 - A_1^2 \; - 4A_0 a_2}{4A_0(b_2 - a_2)}$$

$$F(i) = \frac{4 * 200^2(i - \frac{120}{2 * 200} + 1)^2 - 120^2 \; - 4 * 200 * 125}{4 * 200 * (175 - 125)}$$

$$F(i) = \frac{4 * 200^2(i - \frac{120}{2 * 200} + 1)^2}{4 * 200 * 50} - \frac{120^2 + 4 * 200 * 125}{4 * 200 * 50}$$

$$F(i) = 4(i + 0.7)^2 - 2.86$$

The CDF graph is shown in Figure 18. Simulation result plot also validate the analytical graph as shown in Figure 19.

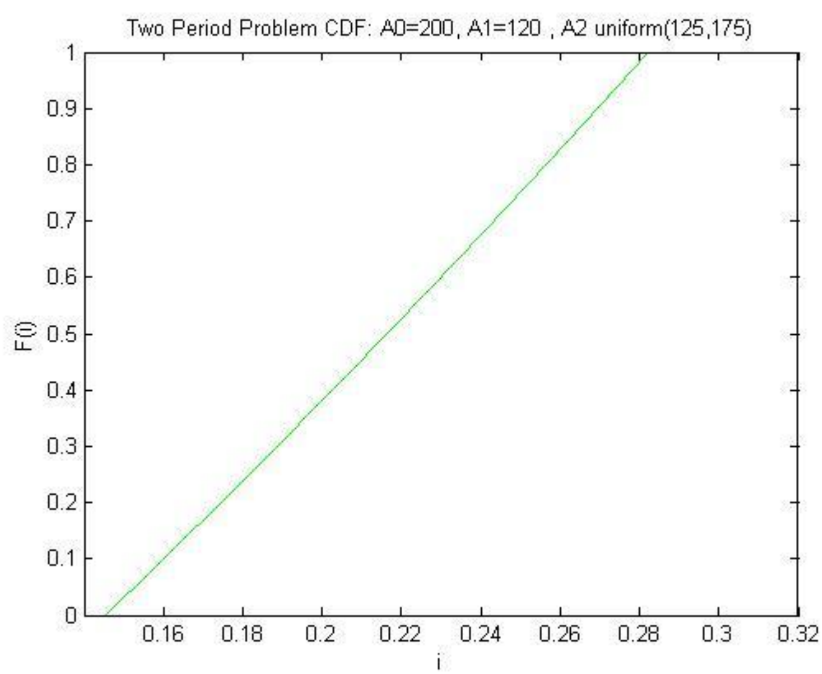

Figure 18 . Two Period Problem CDF Graph: A0=200, A1=120, A2 Uniform (125, 175)

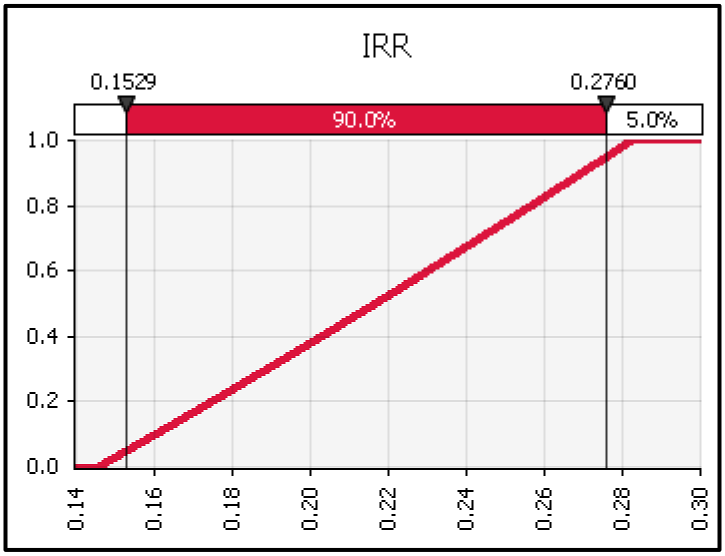

Figure 19 . Two Period Problem Simulation of CDF: A0=200, A1=120, A2 Uniform (125, 175)

Expectation:

$$E(i) = \frac{A_1}{2A_0} - 1 + \frac{\left((A_1^2 + \ 4A_0b_2)^{\frac{3}{2}} - (A_1^2 + \ 4A_0a_2)^{\frac{3}{2}}\right)}{12(b_2 - a_2){A_0}^2}$$

$$E(i) = \frac{120}{2*200} - 1 + \frac{\left((120^2 + \ 4*200*175)^{\frac{3}{2}} - (120^2 + \ 4*200*125)^{\frac{3}{2}}\right)}{12(175-125)200^2} = 0.21567$$

Variance:

$$\sigma^2 = \frac{2A_0}{b_2 - a_2}\left(\frac{i^4}{4} - \frac{i^3}{3}\left(\frac{A_1}{2A_0} - 1\right)\right)\Bigg|_{\frac{\sqrt{A_1^2 + 4A_0a_2}}{2A_0}+\frac{A_1}{2A_0}-1}^{\frac{\sqrt{A_1^2 + 4A_0b_2}}{2A_0}+\frac{A_1}{2A_0}-1} - \left(E(i)\right)^2$$

$$\sigma^2 = \frac{2*200}{175-125}\left(\frac{i^4}{4} - \frac{i^3}{3}\left(\frac{120}{2*200} - 1\right)\right)\Bigg|_{\frac{\sqrt{120^2+4*200*125}}{2*200}+\frac{120}{2*200}-1}^{\frac{\sqrt{120^2+4*200*175}}{2*200}+\frac{120}{2*200}-1} - (0.21567)^2$$

$$\sigma^2 = 8\left(\frac{i^4}{4} + 0.7\frac{i^3}{3}\right)\Bigg|_{0.1455}^{0.2823} - (0.21567)^2 = 0.001537$$

Variance and expectation are the same as the values found by the simulation as shown in Table 14.

Table 14 . Two Period Problem Simulation Statistics: A0=200, A1=120, A2 Uniform (125,175)

Summary Statistics for IRR	
Minimum	0.14558
Maximum	0.28233
Mean	0.21567
Std Dev	0.03945
Variance	0.001556019

4.4 Numerical Example for Two Period IRR Problem: A_0 Constant, A_1 Constant, A_2 Exponential

Let $A_0 = 200,\ A_1 = 120\ and\ A_2\ \sim$ Exponential ($\lambda = 1/175$)

$$\frac{A_1}{A_0} - 1 \leq z \leq \infty$$

$$\frac{120}{200} - 1 \leq z \leq \infty$$

$$-0.4 \leq z \leq \infty$$

$$f(i) = 2\lambda A_0\left(i - \frac{A_1}{2A_0} + 1\right)e^{\left(-\lambda A_0\left(i-\frac{A_1}{2A_0}+1\right)^2+\frac{\lambda A_1^2}{4A_0}\right)}$$

$$f(i) = 2\frac{1}{175}200\left(i - \frac{120}{2*200} + 1\right)e^{\left(-\frac{1}{175}*200\left(i-\frac{120}{2*200}+1\right)^2+\frac{\frac{1}{175}*120^2}{4*200}\right)}$$

$$f(i) = \frac{16}{7}(i + 0.7)e^{\left(-\frac{8}{7}(i+0.7)^2+0.18*175\right)}$$

Figure 20 and Figure 21 below show the graphs of the PDF using MATLAB and @RISK.

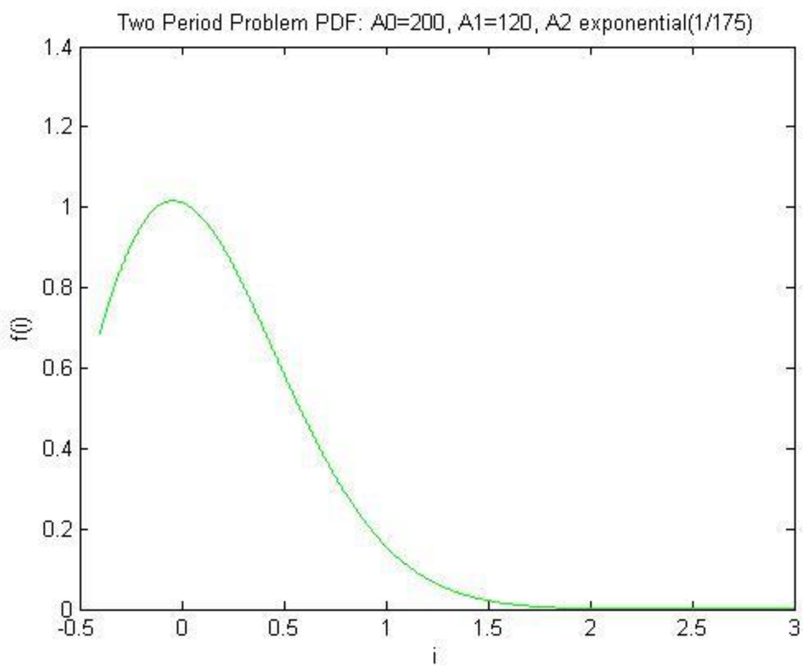

Figure 20 . Two Period Problem PDF Graph: A0=200, A1=120, A2 Exponential (1/175)

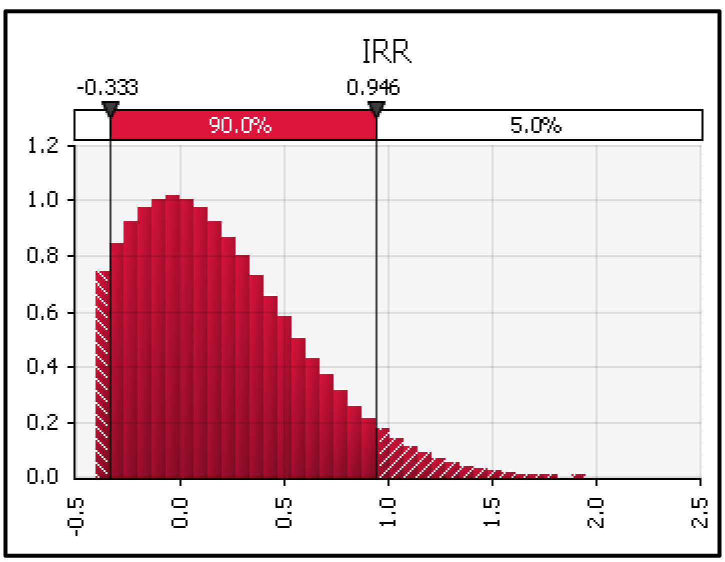

Figure 21 . Two Period Problem Simulation of PDF: A0=200, A1=120, A2 Exponential (1/175)

$$F(i) = 1 - e^{\left(-\lambda A_0\left(i - \frac{A_1}{2A_0} + 1\right)^2 + \frac{\lambda A_1^2}{4A_0}\right)}$$

$$F(i) = 1 - e^{\left(-\frac{1}{175} * 200\left(i - \frac{120}{2*200} + 1\right)^2 + \frac{\frac{1}{175}120^2}{4*200}\right)}$$

$$F(i) = 1 - e^{\left(-\frac{8}{7}(i+0.7)^2 + 0.18*175\right)}$$

The CDF graph is shown in Figure 22 . Simulation result plot also validate the analytical graph as shown in Figure 23.

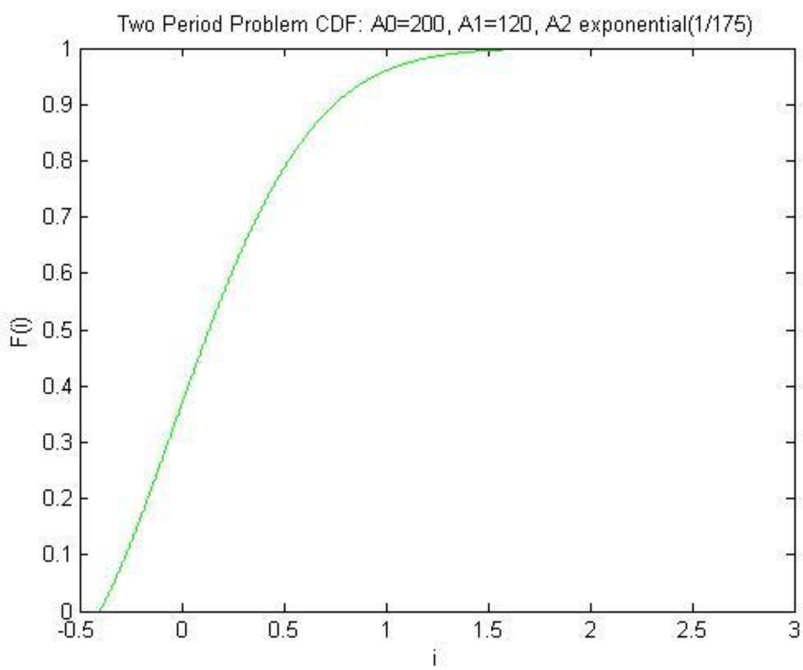

Figure 22 . Two Period Problem CDF Graph: A0=200, A1=120, A2 Exponential (1/175)

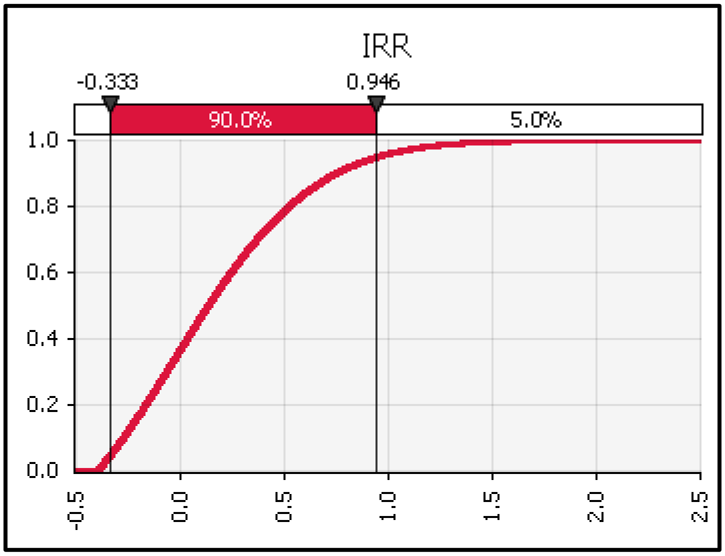

Figure 23 . Two Period Problem Simulation of CDF: A0=200, A1=120, A2 Exponential (1/175)

Expectation:

$$E(i) = \int_{\frac{A_1}{A_0}-1}^{\infty} 2\lambda A_0 \left(i - \frac{A_1}{2A_0} + 1\right) e^{\left(-\lambda A_0 \left(i - \frac{A_1}{2A_0}+1\right)^2 + \frac{\lambda A_1^2}{4A_0}\right)} i\, di =$$

$$E(i) = \int_{\frac{120}{200}-1}^{\infty} 2\frac{1}{175} 200 \left(i - \frac{120}{2*200} + 1\right) e^{\left(-\frac{200}{175}\left(i - \frac{120}{2*200}+1\right)^2 + \frac{\frac{1}{175}120^2}{4*200}\right)} i\, di$$

$$= \int_{-0.4}^{\infty} \frac{16}{7}(i + 0.7) e^{\left(-\frac{8}{7}(i+0.7)^2 + 0.18*175\right)} i\, di$$

$$= -\frac{2}{5} + \frac{1}{8} * \sqrt{14\pi} * e^{\left(\frac{18}{175}\right)} \left(1 - erf\left(\frac{3}{35} * \sqrt{14}\right)\right) = 0.1974$$

Variance:

$$\int_{\frac{A_1}{A_0}-1}^{\infty} 2\lambda A_0 \left(i - \frac{A_1}{2A_0} + 1\right) e^{\left(-\lambda A_0 \left(i - \frac{A_1}{2A_0}+1\right)^2 + \frac{\lambda A_1^2}{4A_0}\right)} i^2\, di - \left(E(i)\right)^2 =$$

$$\int_{\frac{A_1}{A_0}-1}^{\infty} \frac{16}{7}(i + 0.7) e^{\left(-\frac{8}{7}(i+0.7)^2 + 0.18*175\right)} i^2\, di - \left(E(i)\right)^2 =$$

$$\frac{207}{200} + \frac{7}{40} * \sqrt{14\pi} * e^{\left(\frac{18}{175}\right)}\left(erf\left(\frac{3\sqrt{14}}{35}\right) - 1\right) - 0.1974^2 = 0.1597$$

Variance, expectation and the limits are the same as the values found by the simulation as shown in Table 15 below.

Table 15 . Two Period Problem Simulation Statistics: A0=200, A1=120, A2 Exponential (1/175)

Summary Statistics for IRR	
Minimum	-0.39996
Maximum	2.29014
Mean	0.19735
Std Dev	0.39970
Variance	0.159761647

4.5 Numerical Example for Two Period IRR Problem: A_0 Constant, A_1 Uniform, A_2 Constant

Let $A_0 = 200,\ A_1 \sim \text{uniform}(a_1 = 125,\ b_1 = 175)$ and $A_2 = 120$

$$\frac{a_1 + \sqrt{{a_1}^2 + 4A_0A_2}}{2A_0} - 1 \leq i \leq \frac{b_1 + \sqrt{{b_1}^2 + 4A_0A_2}}{2A_0} - 1$$

$$\frac{125 + \sqrt{125^2 + 4 * 200 * 120}}{2 * 200} - 1 \leq i \leq \frac{175 + \sqrt{175^2 + 4 * 200 * 120}}{2 * 200} - 1$$

$$0.1477 \leq i \leq 0.32711$$

$$f(i) = \frac{1}{b_1 - a_1}\left[A_0 + \frac{A_2}{(i+1)^2}\right]$$

$$f(i) = \frac{1}{175 - 125}\left[200 + \frac{120}{(i+1)^2}\right] = \left[4 + \frac{12}{5(i+1)^2}\right]$$

$$F(i) = \frac{A_0(i+1) - \frac{A_2}{(i+1)} - a_1}{b_1 - a_1}$$

$$F(i) = \frac{200(i+1) - \frac{120}{(i+1)} - 125}{175 - 125}$$

$$F(i) = 4(i+1) - \frac{12}{5(i+1)} - 2.5$$

PDF and CDF graphs are in Figure 24 and Figure 25. The simulation plots are in Figure 26 and Figure 27.

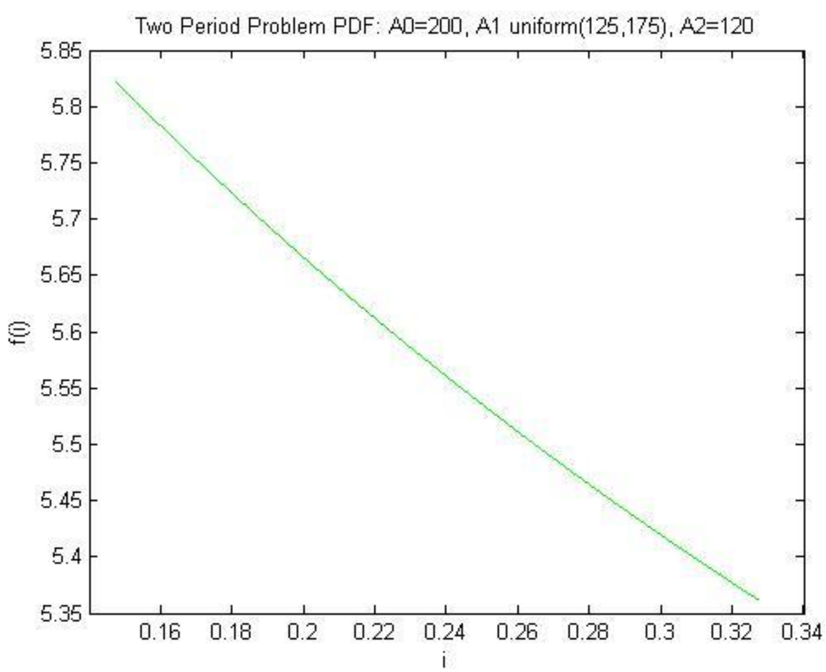

Figure 24 . Two Period Problem PDF Graph: A0=200, A1 Uniform (125, 175), A2=120

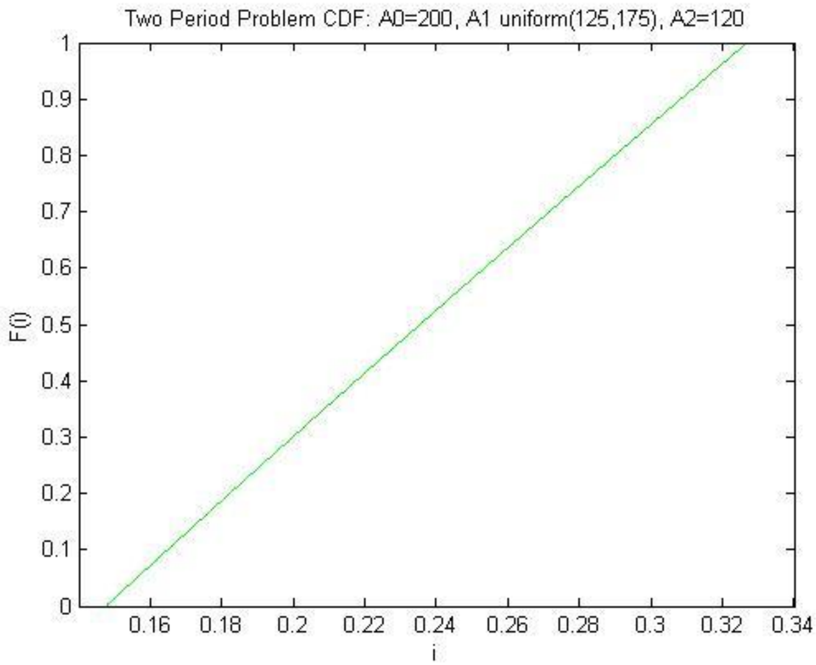

Figure 25 . Two Period Problem CDF Graph: A0=200, A1 Uniform (125, 175), A2=120

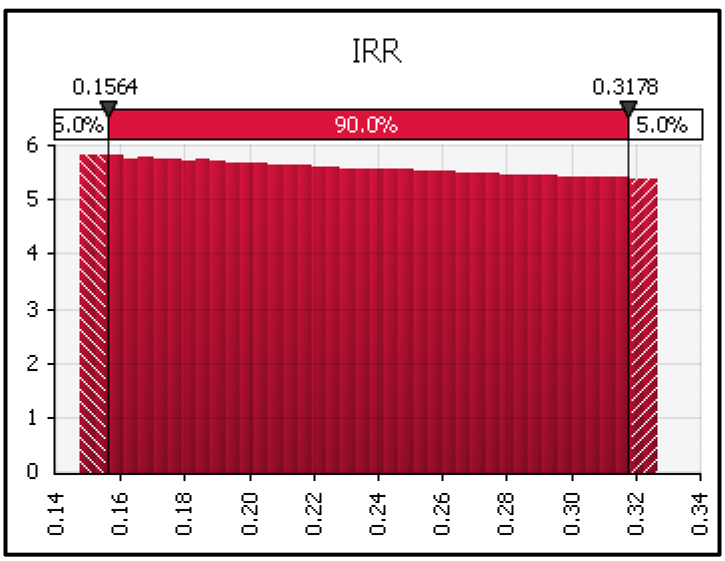

Figure 26 . Two Period Problem Simulation of PDF: A0=200, A1 Uniform (125, 175), A2=120

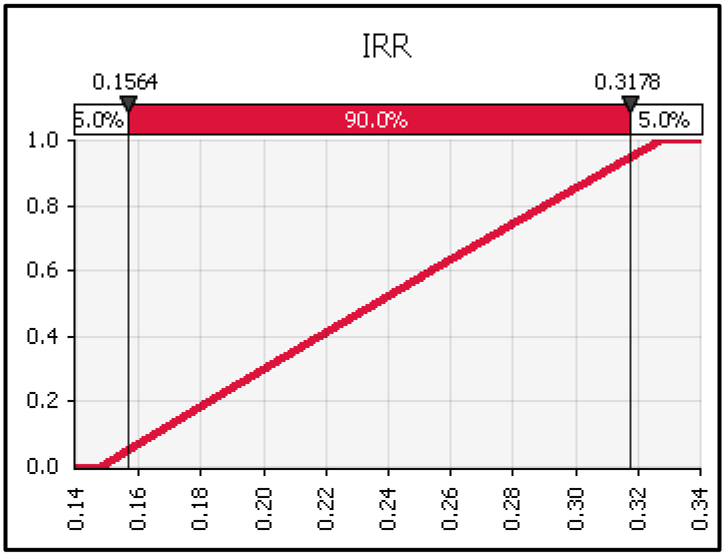

Figure 27 . Two Period Problem Simulation of CDF: A0=200, A1 Uniform (125, 175), A2=120

Expectation:

$$E(i) = \frac{1}{b_1 - a_1}\left[A_0\frac{i^2}{2} + \frac{A_2}{(1+i)} + A_2 ln\,(1+i)\right]\Bigg|_{\frac{a_1+\sqrt{a_1{}^2-4A_0A_2}}{2A_0}-1}^{\frac{b_1+\sqrt{b_1{}^2-4A_0A_2}}{2A_0}-1}$$

$$E(i) = \frac{1}{175-125}\left[200\frac{i^2}{2} + \frac{120}{(1+i)} + 120 ln\,(1+i)\right]\Bigg|_{0.1477}^{0.32711}$$

$$E(i) = \left[4\frac{i^2}{2} + \frac{12}{5(1+i)} + \frac{12}{5} ln\,(1+i)\right]\Bigg|_{0.1477}^{0.32711} = 0.2362$$

Variance:

$$\sigma^2 = \frac{1}{b_1 - a_1}\left[A_0\frac{i^3}{3} + A_2\left(\frac{i(i+2)}{(1+i)} - 2l\,n(1+i)\right)\right]\Bigg|_{\frac{a_1+\sqrt{a_1{}^2-4A_0A_2}}{2A_0}-1}^{\frac{b_1+\sqrt{b_1{}^2-4A_0A_2}}{2A_0}-1} - [E(i)]^2$$

$$\sigma^2 = \frac{1}{175-125}\left[200\frac{i^3}{3} + 120\left(\frac{i(i+2)}{(1+i)} - 2ln(1+i)\right)\right]\Bigg|_{\frac{a_1+\sqrt{{a_1}^2-4A_0A_2}}{2A_0}-1}^{\frac{b_1+\sqrt{{b_1}^2-4A_0A_2}}{2A_0}-1} - [E(i)]^2$$

$$\sigma^2 = \left[4\frac{i^3}{3} + \frac{12}{5}\left(\frac{i(i+2)}{(1+i)} - 2ln(1+i)\right)\right]\Bigg|_{0.1477}^{0.32711} - [0.2362]^2 = 0.00269$$

The summary statistics in Table 16 obtained by using @RISK also verify that the expected value, the variance and the limits of the PDF of IRR are calculated correctly.

Table 16 . Two Period Problem Simulation Statistics: A0=200, A1 Uniform (125, 175), A2=120

Summary Statistics for IRR	
Minimum	0.14776
Maximum	0.32710
Mean	0.23621
Std Dev	0.05179
Variance	0.00268254

4.6 Numerical Example for Two Period IRR Problem: A_0 Constant, A_1 Exponential, A_2 Constant

Let $A_0 = 200,\ A_1 \sim$ exponential $(\lambda = 1/175)$ and $A_2 = 120$

$$\sqrt{\frac{A_2}{A_0}} - 1 \le i \le \infty$$

$$\sqrt{\frac{120}{200}} - 1 \le i \le \infty$$

$$-0.2254 \le i \le \infty$$

$$f(i) = \left(A_0\lambda + \frac{\lambda A_2}{(i+1)^2}\right) e^{\left(-A_0\lambda(i+1) + \frac{\lambda A_2}{(i+1)}\right)}$$

$$f(i) = \left(\frac{200}{175} + \frac{120}{175(i+1)^2}\right) e^{\left(-\frac{200}{175}(i+1) + \frac{120}{175(i+1)}\right)}$$

$$f(i) = \left(\frac{8}{7} + \frac{24}{35(i+1)^2}\right) e^{\left(-\frac{8}{7}(i+1) + \frac{24}{35(i+1)}\right)}$$

The PDF graph obtained by MATLAB is in Figure 28. The simulation result graph is in Figure 29.

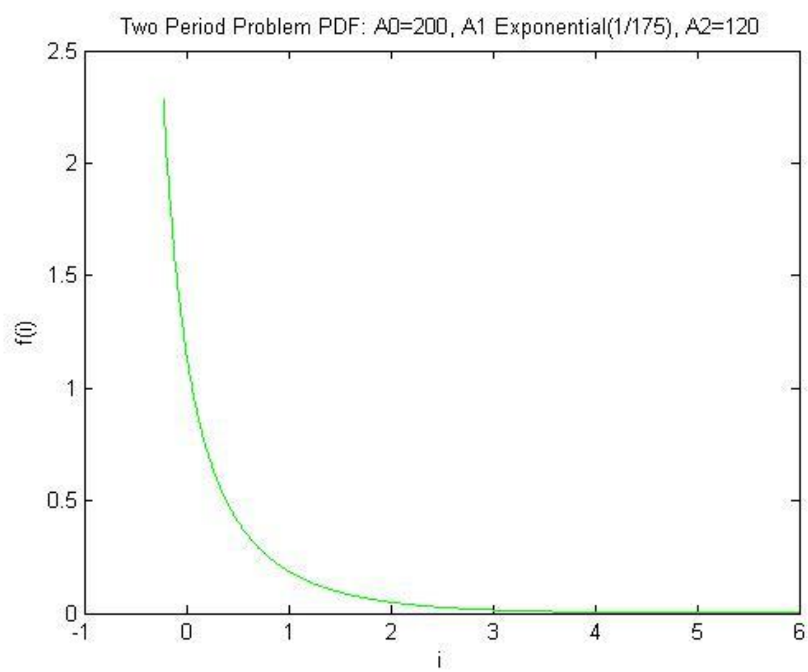

Figure 28 . Two Period Problem PDF Graph: A0=200, A1 Exponential (1/175), A2=120

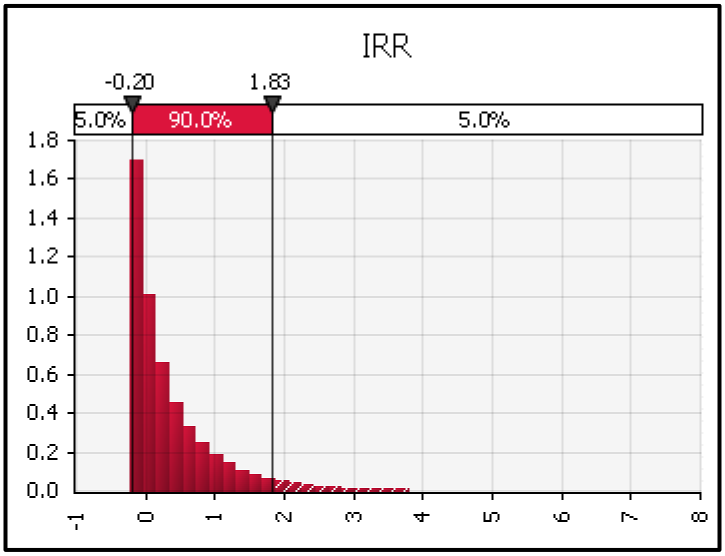

Figure 29. Two Period Problem Simulation of PDF: A0=200, A1 Exponential (1/175), A2=120

Although these two graphs start at the same value of IRR, the corresponding frequencies seems to be different. As simulation yields larger values near infinity, f(i) for the lower limit has a smaller value. Even when the largest 100 iterations are deleted from the simulation results, the frequencies tend to increase f(i) of lower limit (-0.2254) over 2.2. It is therefore logical to assume that the graphs have the same pattern. In addition, the expectation and the variance found later for this case are the same for both the analytical and the simulation results.

The CDF expression is calculated next:

$$F(i) = 1 - e^{\left(-A_0\lambda(i+1)+\frac{\lambda A_2}{(i+1)}\right)}$$

$$F(i) = 1 - e^{\left(-\frac{200}{175}(i+1)+\frac{120}{175(i+1)}\right)}$$

$$F(i) = 1 - e^{\left(-\frac{8}{7}(i+1)+\frac{24}{35(i+1)}\right)}$$

The CDF graphs obtained by MATLAB and @RISK are as shown in Figure 30 and Figure 31.

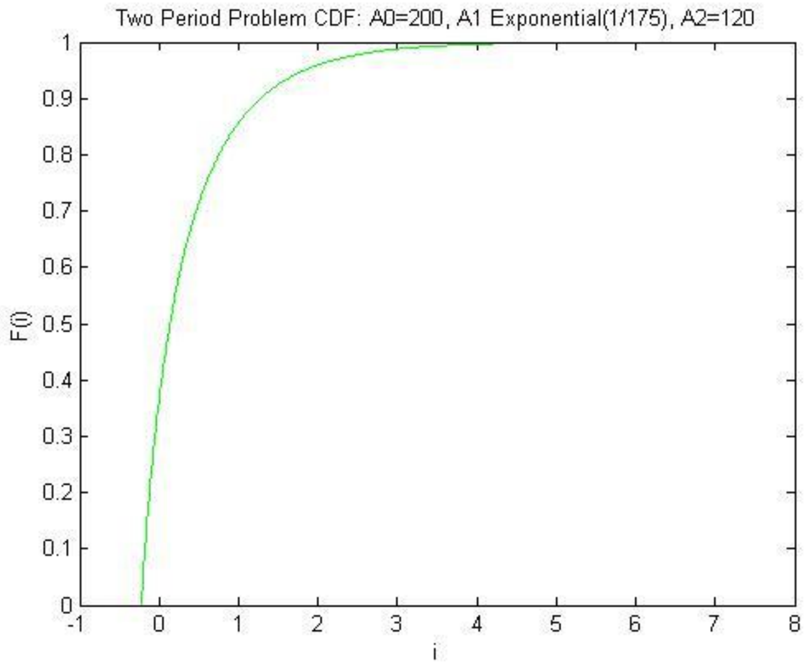

Figure 30 . Two Period Problem CDF Graph: A0=200, A1 Exponential (1/175), A2=120

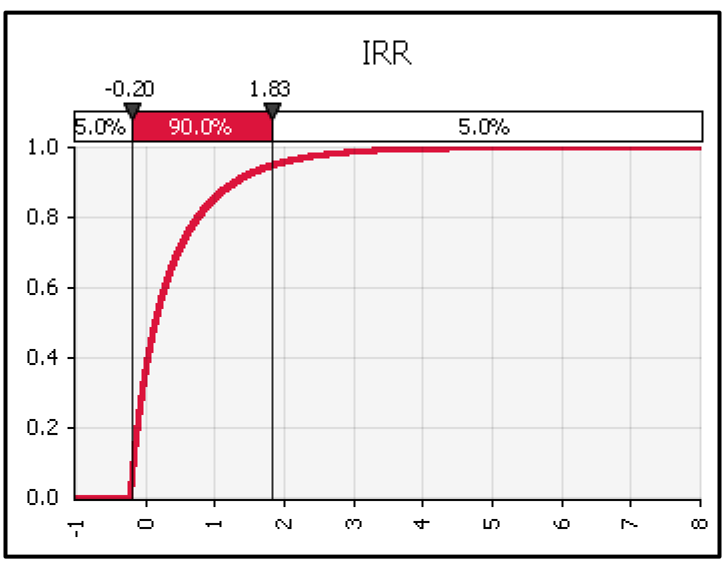

Figure 31 . Two Period Problem Simulation of CDF: A0=200, A1 Exponential (1/175), A2=120

Expectation:

$$E(i) = \int_{\sqrt{\frac{A_2}{A_0}}-1}^{\infty} \left(A_0\lambda + \frac{\lambda A_2}{(i+1)^2}\right) e^{\left(-A_0\lambda(i+1)+\frac{\lambda A_2}{(i+1)}\right)} i \, di$$

$$E(i) = \int_{\sqrt{\frac{120}{200}}-1}^{\infty} \left(\frac{200}{175} + \frac{120}{175(i+1)^2}\right) e^{\left(-\frac{200}{175}(i+1)+\frac{120}{175(i+1)}\right)} i \, di$$

$$= \int_{-0.2254}^{\infty} \left(\frac{8}{7} + \frac{24}{35(i+1)^2}\right) e^{\left(-\frac{8}{7}(i+1)+\frac{24}{35(i+1)}\right)} i \, di = 0.3859$$

Variance:

$$\sigma^2 = \int_{\sqrt{\frac{A_2}{A_0}}-1}^{\infty} \left(A_0\lambda + \frac{\lambda A_2}{(i+1)^2}\right) e^{\left(-A_0\lambda(i+1)+\frac{\lambda A_2}{(i+1)}\right)} i^2 \, di - [E(i)]^2$$

$$= \int_{\sqrt{\frac{120}{200}}-1}^{\infty} \left(\frac{200}{175} + \frac{120}{175(i+1)^2}\right) e^{\left(-\frac{200}{175}(i+1)+\frac{120}{175(i+1)}\right)} i^2 \, di - [E(i)]^2$$

$$= \int_{-0.2254}^{\infty} \left(\frac{8}{7} + \frac{24}{35(i+1)^2}\right) e^{\left(-\frac{8}{7}(i+1)+\frac{24}{35(i+1)}\right)} i^2 \, di - [0.3859]^2 = 0.5194$$

The summary statistics in Table 17 obtained by using @RISK also verify that the expected value, the variance and the limits of the PDF of IRR are calculated correctly.

Table 17 . Two Period Problem Simulation Statistics: A0=200, A1 Exponential (1/175), A2=120

Summary Statistics for IRR	
Minimum	-0.2254
Maximum	7.4642
Mean	0.3859
Std Dev	0.7204
Variance	0.518924824

4.7 Example for the Use of the PDF of IRR: One Period Problem

A short term investment is under consideration. The A_0 and A_1 terms are uniform random variables with parameters $(a_0 = 80,\ b_0 = 100)$ and $(a_1 = 105,\ b_1 = 140)$. Using the mean values, the project has an IRR of 36.11%.

The CDF of IRR is mainly used to evaluate the probability that the IRR will be greater than the MARR. This probability is an effective decision tool in the assessment of the IRR.

$$P(IRR > MARR) = 1 - F(IRR)$$

For this example, the intervals for the IRR are found as the following:

Interval 1:

$$\frac{a_1}{b_0} - 1 \le i < \frac{a_1}{a_0} - 1 \quad \rightarrow \quad \frac{105}{100} - 1 \le i < \frac{105}{80} - 1 \quad \rightarrow \quad 0.05 \le i < 0.3125$$

Interval 2:

$$\frac{a_1}{a_0} - 1 \le i < \frac{b_1}{b_0} - 1 \quad \rightarrow \quad \frac{105}{80} - 1 \le i < \frac{140}{100} - 1 \quad \rightarrow \quad 0.3125 \le i < 0.4$$

Interval 3:

$$\frac{b_1}{b_0} - 1 \le i < \frac{b_1}{a_0} - 1 \quad \rightarrow \quad \frac{140}{100} - 1 \le i < \frac{140}{80} - 1 \quad \rightarrow \quad 0.4 \le i < 0.75$$

For various MARR values within these ranges, the probability that IRR is greater than MARR is calculated. The CDF expressions used in this problem are presented in Table 8.

In the first interval the selected MARR values are 10%, 15%, 20%, 25% and 30%. The following calculation is for 10% . The other MARR values can be calculated similarly.

$$P(IRR > 10\%) = 1 - \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(b_0{}^2(i+1) - 2b_0a_1 + \frac{a_1{}^2}{(i+1)}\right)$$

$$= 1 - \frac{1}{2}\frac{1}{100 - 80}\frac{1}{140 - 105}\left(100^2(0.1+1) - 2*100*105 + \frac{105^2}{(0.1+1)}\right) = 0.9837$$

In the second interval, the MARR value is selected as 35%.

$$P(IRR > 35\%) = 1 - \frac{1}{2}\frac{1}{b_1 - a_1}[(b_0 + a_0)(i+1) - 2a_1]$$

$$= 1 - \frac{1}{2}\frac{1}{140 - 105}((100 + 80)(0.35 + 1) - 2*105) = 0.5285$$

In the third interval, the MARR values are 40% and 45%. The following example is for 45%.

$$P(IRR > 45\%) = 1 - 1 + \frac{1}{2}\frac{1}{b_0 - a_0}\frac{1}{b_1 - a_1}\left(\frac{b_1{}^2}{(i+1)} - 2b_1a_0 + a_0{}^2(i+1)\right)$$

$$= \frac{1}{2}\frac{1}{100 - 80}\frac{1}{140 - 105}\left(\frac{140^2}{(0.45+1)} - 2*140*80 + 80^2(0.45+1)\right) = 0.2837$$

In addition to the CDF method, Hillier's [22] method is also used to find this probability. Hillier [22] uses normal random variables in the calculations. In order to apply his method to this problem, the uniform variables are converted to normal random variables with the same mean but a standard deviation of $\sigma = (b - a)/\sqrt{12}$. So the means and the standard deviations are calculated in the following equations. Equations 7 and 8 are applied to find the mean and the standard deviation of the NPV for MARR 10%.

$$\mu_1 = \frac{80 + 100}{2} = 90 \qquad \mu_2 = \frac{105 + 140}{2} = 122.5$$

$$\sigma_1 = \frac{100 - 80}{\sqrt{12}} = 5.7735 \qquad \sigma_2 = \frac{140 - 105}{\sqrt{12}} = 10.1036$$

$$\mu_p = \sum_{j=0}^{1} \left[\frac{E(Y_j)}{(1+i)^j}\right] = -90 + \frac{122.5}{1.1} = 21.36$$

$$\sigma_p^2 = \sum_{j=0}^{1} \left[\frac{Var(Y_j)}{(1+i)^{2j}}\right] = 5.7735^2 + \frac{10.1036^2}{1.1^2} = 117.7$$

Then the cumulative normal distribution table is used to find the probability of that NPV is less than zero.

$$Prob(NPV < 0|MARR = 10\%) = P\left(\frac{0 - 21.36}{\sqrt{117.7}}\right) = 0.0245$$

$$Prob(IRR > 10\%) = 1 - 0.0245 = 0.9755$$

Table 18 shows all of the probability results with the CDF method and Hillier's method. It can be seen that the probabilities found by each method are really very close to each other.

Table 18 . P (IRR>MARR) Calculations in One Period Problem

MARR	P(IRR>MARR)	Hiller's Method
10%	0.9837	0.9755
15%	0.9378	0.9419
20%	0.8660	0.8817
25%	0.7714	0.7897
30%	0.6565	0.6689
35%	0.5285	0.5312
40%	0.4000	0.3933
45%	0.2837	0.2710

4.8 Example for the Use of the PDF of IRR: Two Period Problem

In this example, a two period problem is considered where the cash flow of the second period is a uniform random variable. The A_0 and A_1 terms are deterministic values, 300 and 150 respectively. The A_2 term is distributed uniformly with the parameters $(a_2 = 200, b_2 = 250)$. Using the deterministic values for the first two cash flows and the mean of the last cash flow, the project has an IRR of 15.14%.

For this problem, the interval IRR is defined is

$$\frac{\sqrt{A_1^2 + 4A_0 a_2}}{2A_0} + \frac{A_1}{2A_0} - 1 \leq i \leq \frac{\sqrt{A_1^2 + 4A_0 b_2}}{2A_0} + \frac{A_1}{2A_0} - 1$$

Plugging in the numerical values:

$$\frac{\sqrt{150^2 + 4 * 300 * 200}}{2 * 300} + \frac{150}{2 * 300} - 1 \leq i \leq \frac{\sqrt{150^2 + 4 * 300 * 250}}{2 * 300} + \frac{150}{2 * 300} - 1$$

$$0.1039 \leq i \leq 0.1964$$

The CDF expression of this example is as in equation 47.

$$F(i) = \frac{4{A_0}^2(i - \frac{A_1}{2A_0} + 1)^2 - A_1^2 - 4A_0 a_2}{4A_0(b_2 - a_2)}$$

The possible MARR values chosen for this example are 11%, 12%, 13%, 14%, 15%, 16%, 17%, 18% and 19%. The following calculation is for 11%. Calculations for the other MARR values can be solved similarly by only changing the *i* value.

$$P(IRR > 11\%) = 1 - \frac{4{A_0}^2(i - \frac{A_1}{2A_0} + 1)^2 - A_1^2 - 4A_0 a_2}{4A_0(b_2 - a_2)}$$

$$= 1 - \frac{4 * 300^2 * (0.11 - \frac{150}{2 * 300} + 1)^2 - 150^2 - 4 * 300 * 200}{4 * 300 * (250 - 200)} = 0.9374$$

Hillier's [22] method can also be applied to find these probabilities. Again, the conversion to random variables should be made in this example. The deterministic values of the A_0 and A_1 terms are assumed to be their means. Their standard deviations are assumed to be zero as they are deterministic values. The uniform A_2 variable has a mean of $(b + a)/2$ and a standard deviation of $(b - a)/\sqrt{12}$ when it is redefined as a normal random variable. The calculations are shown below.

$$\mu_1 = 300 \qquad \mu_2 = 150 \qquad \mu_3 = \frac{200 + 250}{2} = 225$$

$$\sigma_1 = 0 \qquad \sigma_2 = 0 \qquad \sigma_3 = \frac{250 - 200}{\sqrt{12}} = 14.4337$$

The mean and the variance of the NPV are found by the following calculations.

$$\mu_p = \sum_{j=0}^{2} \left[\frac{E(Y_j)}{(1 + i)^j}\right] = -300 + \frac{150}{1.11} + \frac{225}{1.11^2} = 17.75$$

$$\sigma_p^2 = \sum_{j=0}^{2} \left[\frac{Var(Y_j)}{(1 + i)^{2j}}\right] = 0^2 + \frac{0^2}{1.11^2} + \frac{14.4337^2}{1.11^4} = 137.23$$

The cumulative normal distribution table is used to find the probability that NPV is less than zero. P (IRR>MARR) is then calculated by subtracting this probability from one.

$$Prob(NPV < 0|MARR = 11\%) = P\left(\frac{0 - 17.75}{\sqrt{137.23}}\right) = 0.0649$$

$$Prob(IRR > 11\%) = 1 - 0.0648 = 0.9351$$

The same calculations are made for all the other MARR values selected. The results of both the CDF and Hiller's method are shown as in Table 19. It can easily be seen that approximation to Hiller's method gives very close results to the real probabilities found by the CDF method.

Table 19 . P (IRR>MARR) Calculations in Two Period Problem

MARR	P(IRR>MARR)	Hiller's Method
11%	0.9374	0.9351
12%	0.8336	0.8760
13%	0.7286	0.7857
14%	0.6224	0.6642
15%	0.5150	0.5207
16%	0.4064	0.3728
17%	0.2966	0.2405
18%	0.1856	0.1380
19%	0.0734	0.0697

Chapter 5 – Conclusions and Recommendations

This thesis has addressed an old engineering economy problem from a different perspective. One and two period engineering economy problems with some random cash flows were considered in order to develop the PDF expressions for the IRR under some limiting assumptions.

One of the major limiting assumptions is about the signs of the cash flows. In all the cases considered the initial investment is assumed to be a cash outflow and the subsequent ones are cash inflows. This is a straight forward assumption in order to have only one valid root for the IRR. In case there is more than one root, this means there are more than one sign changes in the problem. As a further study, it can be suggested that the projects can be chosen as nonsimple investments where there can be cash outflows in any period of the project. Then the roots of IRR should be determined initially and the selection of the valid IRR should be made. The PDF of IRR should be obtained by the distribution function method similar to the ones used in this thesis.

Another major classification is obtained by determination of the random cash flows in the problems. For one period problem, two different cases considered construct all the possible combinations. The A_0 term is assumed to be a constant or a random variable at a time whereas the A_1 term is always a random variable. For the two period problem, A0 is always assumed to be a constant variable for the ease of the computations. For the two different cases considered, either A_1 or A_2 are assumed to be random variables at a time while the remaining cash flows have deterministic values.

In all the cases for both one and two period problems, distribution function method gives a result as the CDF formula can be derived directly. However, in two period problem, the IRR root equation needs square, square root, sum and quotient transformations when the transformation of random variables method is used. Even if there is only one random variable in the problem, some kind of correlation has to be determined between the sum and the square root transformations. Therefore, the case where only A_1 is a random variable in two period problem, the derivation of the PDF of IRR is maintained by only the distribution function method. It may be actually possible to determine the transformation of the IRR equation for this case but in the context of this thesis, this way could not be calculated.

So as a further study, the PDF of IRR for two more cases should be determined. In the first case, both A_1 and A_2 should be taken as random variables. This problem can also be solved by distribution of random variables method. If transformation of random variables method is used, then this problem requires sum, power and square root transformations of random variables in order to find the distribution of the IRR. The second case should consider all the cash flows random and therefore derivation of the exact PDF expression for the IRR or *i* requires the sum, quotient, and power of one or more of these three random cash flows. This problem can also be solved by the distribution of random variables method easier. In both of these cases, as there is more than one random variable, they can be assumed to be independent random variables.

The cash flows considered in this thesis or the ones suggested for further research can also be assumed have correlation between each other. The correlation coefficients should be determined and included in formulation of the problem. This will yield a more complicated problem as the correlations would be defined according to the distributions selected for the random variables. The distributions used in this study are the exponential and uniform distributions. It may be worth to consider other distributions, such as normal, with the same cases. However, it should be noted that the closed form expressions for the PDF of IRR would be derived after the intervals where IRR exists are determined initially. Therefore, it is important to select the distributions where the extreme limits of the distributions could be easily modified for the IRR intervals. Joint distributions for the random cash flows can also be studied as further research. For the two period problem, the A1 and A2 terms can be assumed to follow a joint distribution and the density of the IRR equation can be derived.

The PDF of IRR problem can also be extended for a 3 period problem. The IRR equation will then be a third degree polynomial instead of a quadratic function. Distribution function method is likely to be used in this problem as well.

Bibliography

[1] Bean, M.A., *Probability: The Science of Uncertainty with Applications to Investments, Insurance, and Engineering*, Brooks/Cole, Pacific Grove, CA, 2001.

[2] Bernhard, R.H., "Probability and Rates of Return: Some Critical Comments", Management Science, 13, 598-600, 1967.

[3] Bower, R.S. and Lessard, D.R., "An Operational Approach to Risk-Screening", Journal of Finance, 28, 321-337, 1973.

[4] Bussey, L.E., *The Economic Analysis of Industrial Projects,* Prentice-Hall, Englewood Cliffs, NJ, 1978.

[5] Carmichael, D.G. and Balatbat, M.C.A., "Probabilistic DCF Analysis and Capital Budgeting and Investment – A Survey", The Engineering Economist, 53, 84-102, 2008.

[6] Casella, G. and Berger, R.L., *Statistical Inference,* 2nd edition, Duxbury, Pacific Grove, CA, 2002.

[7] Curtis, J.H., "On the Distribution of the Quotients of Two Chance Variables", Annals of Mathematical Statistics, 12, 409-421, 1941.

[8] DeGarmo, E.P., Sullivan, W.G. and Bontadelli, J.A., *Engineering Economy,* 8th edition, Macmillan Publishing Company, NY, 1989.

[9] DeGroot, M.H. and Schervish M.J., *Probability and Statistics*, 3rd edition, Addison-Wesley, USA, 2002.

[10] Ekern, S., "On the Inadequacy of a Probabilistic Internal Rate of Return", Journal of Business Finance and Accounting, 6, 2, 229-237, 1979.

[11] Elnicki, R.A., "ROI simulations for Investment Decisions", Management Accounting, 51, February, 37-41, 1970.

[12] Erdos, P. and Turan, P., "On the Distribution of Roots of Polynomials", Annals of Mathematics, 51, 1, 105-119, 1950.

[13] Fairley, W. and Jacoby, H. D., "Investment Analysis Using the Probability Distribution of the Internal Rate of Return", Management Science, 21, 12, 1428-1437, 1975.

[14] Fieller, E.C., "Some Problems in Interval Estimation", Journal of the Royal Statistical Society, B, 16, 175-184, 1954.

[15] Fieller, E.C., "The Distribution of the Index in a Normal Bivariate Population", Biometrika, 24, 428-440, 1932.

[16] Geary, R.C., "The Frequency Distribution of the Quotient of Two Normal Variates", Journal of the Royal Statistical Society, 93, 442-446, 1930.

[17] Giffin, W.C., *Transform Techniques for Probability Modeling*, Academic Press, NY, 1975.

[18] Hayya, J., et al., "A Note on the Ratio of Two Normally Distributed Variables", Management Science, 21, 11, 1338-1341, 1975.

[19] Hertz, D.B., "Risk Analysis and Capital Investment", Harvard Business Review, 42, 1, 95-106, 1964.

[20] Hamblen, W., "Distribution of Roots of Quadratic Equations with Random Coefficients", Annals of Mathematical Statistics, 27, 1136--1143, 1956.

[21] Heinen, M. E., "The IRR Problem", http://www.markeredwards.com, CSU Projects

on left pane.
[22] Hillier, F., S., "The Derivation of Probabilistic Information for the Evaluation of Risky Investments", Management Science, 9, 3, 443-457, 1963.
[23] Hillier, F.S., "Supplement to 'The Derivation of Probabilistic Information for the Evaluation of Risky Investments' ", Management Science, 11, 3, 485-487, 1965.
[24] Hinkley, D.V., "On the Ratio of Two Correlated Normal Random Variables", Biometrika, 56, 3, 635, 1969.
[25] Hodges, S.D. and Moore, P.G., "The Consideration of Risk in Project Selection", Journal of the Institute of Actuaries, 94, 355-378, 1968.
[26] Kahraman, C., Tolga, E., and Ulukan, Z., "Probabilistic Cash Flows and Equivalent Discount Rates", Proceedings of the IEEE International Conference on Systems, Man and Cybernetics, 2, 634-637, 1999.
[27] Kaplan, S. and Barish, N.N., "Decision-Making Allowing Uncertainty of Future Investment Opportunities", Management Science, 13, 10, B-569-577, 1967.
[28] Keeley, R.H. and Westerfield, "A Problem in Probability Distribution Techniques in Capital Budgeting", Journal of Finance, 27, 703-709, 1972.
[29] Lewellen, W.G. and Long, M.S., "Simulation versus Single-Value Estimates In Capital Expenditure Analysis", Decision Sciences, 3, 19-33, 1972.
[30] Marsaglia, G., "Ratios of Normal Variables and Ratios of Sums of Uniform Variables", Journal of the American Statistical Association, 60, 309, 193-204, 1965.
[31] Moore, W. T. and Chen, S., "Implementing the IRR Criterion When the Cash Flow Parameters are Unknown", Financial Review, 19, 4, 351-358, 1984.
[32] Palisade Corp., *@RISK, Advanced Risk Analysis for Spreadsheets*, Newfield, N.Y., 2004.
[33] Park, C.S. and Gunter, P. S., *Advanced Engineering Economics*, J. Wiley and Sons, NY, NY, 1990.
[34] Pohjola, V.J. and Turunen, I., "Estimating the Internal Rate of Return from Fuzzy Data", Engineering Costs and Production Economics, 18, 215-221, 1990.
[35] Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, 2nd edition, McGraw-Hill, USA, 1984.
[36] Ranasinghe, M. and Russell, A.D., "Analytical Approach for Economic Risk Quantification of Large Engineering Projects: Validation", Construction Management and Economics, 10, 45-68, 1992.
[37] Renwick, F. B., *Introduction to Investment Analysis and Finance*, Macmillan, 1971.
[38] Robichek, A.A., "Interpreting the Results of Risk Analysis", Journal of Finance, 30, 1384-1386, 1975.
[39] Rohatgi, V.K., *Statistical Inference*, Dover Publications, Mineola, NY, 2003.
[40] Rothkopf, M.H., "A Substitute for Expected Rate of Return", Management Science, 12, 1, 148-149, 1965.
[41] Sarper, H., "Revisit of Newton's Method for the Internal Rate of Return (IRR) Problem", Journal of Taxation of Investments, 362-368, Summer 2003.
[42] Schriber, T.J., "Simulation of Probabilistic Cash Flows and Internal Rates of Flow", *FORTRAN Case Studies for Business Applications,* John Wiley and Sons Inc., NY, NY, 1969.

[43] Shooman, M.L., *Probabilistic Reliability: An Engineering Approach*, 2nd edition, R.E. Krieger Publishing Company, Malabar, FL, 1990.

[44] Transport Economics, Policy and Poverty Thematic Group, *Transport Notes*, No. TRN-7, The World Bank, Washington DC., 2005.

[45] Wagle, B., "A Statistical Analysis of Risk in Capital Investment Projects", Operations Research Quarterly, 18, 1, 13-33, 1967.

Appendices

APPENDIX A

Fred Blackwell Renwick explains the definition and calculation of IRR in the book Introduction to Investments and Finance (1971) [37] as the following:

“ **Criterion of Internal Rate of Return**

Computationally, internal rate of return is similar to net present value. Both methods use the same formula, equation (5.1).

$$P = \frac{A_0}{(1+k)^0} + \frac{A_1}{(1+k)^1} + \frac{A_2}{(1+k)^2} + \cdots + \frac{A_n}{(1+k)^n}$$

Instead of assuming a value for *k* and using this value to calculate *P*, a known value for *P* is given and then equation is solved for *k*. Equation (5.1)., therefore, is the general form useful in calculating either net present value or internal rate of return. Conceptually, however, internal rate of return is quite different from net present value. There are two ways to view the internal rate of return. Both ways yield identical results.

First, instead of denoting total dollars of net profit (as does the net present value criterion), internal rate of return denotes total profits expressed as a percent of total investment outlays. As the name implies, IRR measures the *average rate of return* received on total outlay *(P)*. All dollar sums in the various cash flows are discounted (or compounded) to the same point in time (usually the present).

To illustrate the first view of IRR as an average rate of return received on investment, consider an elementary case. If a government bond is purchased today for $1000 and sold 1 year hence at the face value of $1000, then after receiving interest income of $60, the internal rate of return (the average rate of return on total investment of $1000) is 6 percent. The $60 profit equals the rate of return multiplied by total present value of investment outlays.

$$\$1000 = \frac{\$1060}{1+k}$$

$$k = \left(\frac{\$1060}{\$1000}\right) - 1 = 0.06$$

The second way to view internal rate of return is equate IRR with the required rate of interest on a savings account – to view the cash outlays associated with the proposed investment as lump sum from the investment project as withdrawals from that savings account. Then the internal rate of return is the rate of interest on a savings account required to provide a balance in the account (i.e., after all deposits and withdrawals) of precisely zero as of the end of the time period under consideration. Other things being equal, the higher the rate of interest, the larger the withdrawals for a given lump sum deposit and the more desirable the account. Ceteris paribus, larger IRR's are superior to smaller IRR's.

To illustrate the second view of IRR, consider the same bond as above. Instead of investing the $1000 in the bond, if the $1000 were to be deposited into a savings account, then what is the rate of interest required on the account to permit the owner to withdraw a total of $1060 (coupon plus face value) as of 1 year hence? The answer, 6 percent, is the required return. The calculations are

$$\$1000\ (1+k)^1 = \$1060$$

$$k = \left(\frac{\$1060}{\$1000}\right) - 1 = 0.06$$

”

APPENDIX B

MATLAB CODES

The following MATLAB codes are used to graph the PDF and CDF expressions for each of the cases.

One Period Problem PDF: A_0 Constant, A_1 Uniform

```
function y=f1(N)
A_0=120;
a_1=125;
b_1=175;

L=(a_1/A_0)-1;
U=(b_1/A_0)-1;
x=linspace(L,U,N);
y=A_0/(b_1-a_1);
plot(x,y,'g-');
```

One Period Problem CDF: A_0 Constant, A_1 Uniform

```
function y=f1a(N)
A_0=120;
a_1=125;
b_1=175;

L=(a_1/A_0)-1;
U=(b_1/A_0)-1;
x=linspace(L,U,N);
y=((x+1)*A_0-a_1)/(b_1-a_1);
plot(x,y,'g-');
```

One Period Problem PDF: A_0 Uniform, A_1 Uniform

Interval 1:

```
function y=f2(N)
a_0=80;
b_0=100;
a_1=120;
b_1=200;

L1=(a_1/b_0)-1;
U1=(a_1/a_0)-1;
```

```
x=linspace(L1,U1,N);
y=((b_0^2)-(a_1^2)./((x+1).^2))./(2*(b_0-a_0)*(b_1-a_1));
plot(x,y,'g-');
```

Interval 2:

```
function y=f2a(N)
a_0=80;
b_0=100;
a_1=120;
b_1=200;

L2=(a_1/a_0)-1;
U2=(b_1/b_0)-1;
x=linspace(L2,U2,N);
y=(b_0+a_0)./(2*(b_1-a_1));
plot(x,y,'g-');
```

Interval 3:

```
function y=f2b(N)
a_0=80;
b_0=100;
a_1=120;
b_1=200;

L3=(b_1/b_0)-1;
U3=(b_1/a_0)-1;
x=linspace(L3,U3,N);
y=((a_0^2)-(b_1^2)./((x+1).^2))./(-2*(b_0-a_0)*(b_1-a_1));
plot(x,y,'g-');
```

One Period Problem CDF: A_0 Uniform, A_1 Uniform

Interval 1:

```
function y=f2cdf(N)
a_0=80;
b_0=100;
a_1=120;
b_1=200;

L1=(a_1/b_0)-1;
```

```
U1=(a_1/a_0)-1;
x=linspace(L1,U1,N);
y=((b_0^2.*(x+1))-2*b_0*a_1+(a_1^2)./(x+1))./(2*(b_0-a_0)*(b_1-a_1));
plot(x,y,'g-');
```

Interval 2:

```
function y=f2cdfa(N)
a_0=80;
b_0=100;
a_1=120;
b_1=200;

L2=(a_1/a_0)-1;
U2=(b_1/b_0)-1;
x=linspace(L2,U2,N);
y=((b_0+a_0).*(x+1)-2*a_1)./(2*(b_1-a_1));
plot(x,y,'g-');
```

Interval 3:

```
function y=f2cdfb(N)
a_0=80;
b_0=100;
a_1=120;
b_1=200;

L3=(b_1/b_0)-1;
U3=(b_1/a_0)-1;
x=linspace(L3,U3,N);
y=1+((a_0^2).*(x+1)+(b_1^2)./(x+1)-2*b_1*a_0)./(-2*(b_0-a_0)*(b_1-a_1));
plot(x,y,'g-');
```

Two Period Problem PDF: A_0 Constant, A_1 Constant, A_2 Uniform

```
function y=f3(N)
A_0=200;
A_1=120;
a_2=125;
b_2=175;

L=A_1/(2*A_0)-1+ ( sqrt(A_1^2+ 4*A_0*a_2 ))/(2*A_0 );
```

```
U=sqrt(A_1^2+ 4*A_0* b_2 )/(2*A_0)+A_1/(2*A_0)-1;
x=linspace(L,U,N);
y=((2*A_0)./(b_2-a_2)).*(x-A_1/(2*A_0)+1);
plot(x,y,'g-');
```

Two Period Problem CDF: A_0 Constant, A_1 Constant, A_2 Uniform

```
function y=f3cdf(N)
A_0=200;
A_1=120;
a_2=125;
b_2=175;

L=A_1/(2*A_0)-1+ ( sqrt(A_1^2+ 4*A_0*a_2 ))/(2*A_0 );
U=sqrt(A_1^2+ 4*A_0* b_2 )/(2*A_0)+A_1/(2*A_0)-1;
x=linspace(L,U,N);
y=(4*A_0^2.*(x-A_1/(2*A_0)+1).^2-A_1^2-4*A_0*a_2)./((4*A_0).*(b_2-a_2));
plot(x,y,'g-');
```

Two Period Problem PDF: A_0 Constant, A_1 Constant, A_2 Exponential

```
function y=f4(N)
A_0=200;
A_1=120;
lambda=1/175;

L=(A_1/A_0)-1;
U=3;
x=linspace(L,U,N);
y=(2*lambda*A_0).*(x-((A_1)./(2*A_0))+1).*exp((-lambda*A_0*(x-
((A_1)./(2*A_0))+1).^2+(lambda*A_1^2)/(4*A_0)));
plot(x,y,'g-');
```

Two Period Problem CDF: A_0 Constant, A_1 Constant, A_2 Exponential

```
function y=f4cdf(N)
A_0=200;
A_1=120;
lambda=1/175;
```

```
L=(A_1/A_0)-1;
U=3;
x=linspace(L,U,N);
y=1-exp((-lambda*A_0.*(x-
((A_1)./(2*A_0))+1).^2)+((lambda*A_1^2)/(4*A_0)));
plot(x,y,'g-');
```

Two Period Problem PDF: A_0 Constant, A_1 Uniform, A_2 Constant

```
function y=f5a(N)
a_0=200;
a_1=125;
b_1=175;
a_2=120;

L1=(a_1+sqrt(a_1^2+4*a_0*a_2))/(2*a_0)-1;
U1=(b_1+sqrt(b_1^2+4*a_0*a_2))/(2*a_0)-1;
x=linspace(L1,U1,N);
y=(a_0+a_2./((x+1).^2))./(b_1-a_1);
plot(x,y,'g-');
```

Two Period Problem CDF: A_0 Constant, A_1 Uniform, A_2 Constant

```
function y=f5b(N)
a_0=200;
a_1=125;
b_1=175;
a_2=120;

L1=(a_1+sqrt(a_1^2+4*a_0*a_2))/(2*a_0)-1;
U1=(b_1+sqrt(b_1^2+4*a_0*a_2))/(2*a_0)-1;
x=linspace(L1,U1,N);
y=(a_0.*(x+1)-a_2./(x+1)-a_1)./(b_1-a_1);
plot(x,y,'g-');
```

Two Period Problem PDF: A_0 Constant, A_1 Exponential, A_2 Constant

```
function y=f6(N)
A_0=200;
lambda=1/175;
```

```
A_2=120;

L=sqrt(A_2/A_0)-1;
U=6;
x=linspace(L,U,N);
y=lambda.*(A_0+A_2./((x+1).^2)).*exp((-
lambda*A_0.*(x+1))+(lambda*A_2./(x+1)));
plot(x,y,'g-');
```

Two Period Problem CDF: A_0 Constant, A_1 Exponential, A_2 Constant

```
function y=f6cdf(N)
A_0=200;
lambda=1/175;
A_2=120;

L=sqrt(A_2/A_0)-1;
U=8;
x=linspace(L,U,N);
y=1-exp((-lambda*A_0.*(x+1))+(lambda*A_2./(x+1)));
plot(x,y,'g-');
```

Vita