Course at a Glance

Plan

The Course at a Glance provides a useful visual organization for the AP Precalculus curricular components, including:

- Sequence of units, along with approximate weighting and suggested pacing. Please note, pacing is based on 45-minute class periods, meeting five days each week for a full academic year.
- Progression of topics within each unit.

Teach

MATHEMATICAL PRACTICES

1 Procedural and Symbolic Fluency

3 Communication and Reasoning

2 Multiple Representations

Required Course Content

Each topic contains required Learning Objectives and Essential Knowledge Statements that form the basis of the assessment on the AP Exam.

Assess

Assign the Progress Checks—either as homework or in class—for each unit. Each Progress Check contains formative multiple-choice and free-response questions. The feedback from the Progress Checks shows students the areas where they need to focus.

UNIT 1 Polynomial and Rational Functions

6–8 weeks | **30–40%** AP Exam Weighting

2 3	**1.1**	Change in Tandem
2 3	**1.2**	Rates of Change
3	**1.3**	Rates of Change in Linear and Quadratic Functions
2 3	**1.4**	Polynomial Functions and Rates of Change
1 2	**1.5**	Polynomial Functions and Complex Zeros
3	**1.6**	Polynomial Functions and End Behavior
1 3	**1.7**	Rational Functions and End Behavior
1	**1.8**	Rational Functions and Zeros
2	**1.9**	Rational Functions and Vertical Asymptotes
3	**1.10**	Rational Functions and Holes
1 3	**1.11**	Equivalent Representations of Polynomial and Rational Expressions
1 3	**1.12**	Transformations of Functions
2 3	**1.13**	Function Model Selection and Assumption Articulation
1 3	**1.14**	Function Model Construction and Application

Progress Check Unit 1 Part 1: Topics 1.1–1.6

Multiple-choice: 18
Free-response: 2

Progress Check Unit 1 Part 2: Topics 1.7–1.14

Multiple-choice: 24
Free-response: 2

UNIT 2 Exponential and Logarithmic Functions

6–9 weeks | **27–40%** AP Exam Weighting

1 3	**2.1**	Change in Arithmetic and Geometric Sequences
1 3	**2.2**	Change in Linear and Exponential Functions
3	**2.3**	Exponential Functions
1 3	**2.4**	Exponential Function Manipulation
1 3	**2.5**	Exponential Function Context and Data Modeling
2 3	**2.6**	Competing Function Model Validation
1 2	**2.7**	Composition of Functions
1 2	**2.8**	Inverse Functions
1	**2.9**	Logarithmic Expressions
1 2	**2.10**	Inverses of Exponential Functions
3	**2.11**	Logarithmic Functions
1 3	**2.12**	Logarithmic Function Manipulation
1	**2.13**	Exponential and Logarithmic Equations and Inequalities
1 3	**2.14**	Logarithmic Function Context and Data Modeling
2 3	**2.15**	Semi-log Plots

Progress Check Unit 2 Part 1: Topics 2.1–2.8

Multiple-choice: 24
Free-response: 2

Progress Check Unit 2 Part 2: Topics 2.9–2.15

Multiple-choice: 24
Free-response: 2

UNIT 3 Trigonometric and Polar Functions

7–10 weeks | **30–35%** AP Exam Weighting

2 3	**3.1**	Periodic Phenomena
2 3	**3.2**	Sine, Cosine, and Tangent
2 3	**3.3**	Sine and Cosine Function Values
2 3	**3.4**	Sine and Cosine Function Graphs
2 3	**3.5**	Sinusoidal Functions
1 2	**3.6**	Sinusoidal Function Transformations
1 3	**3.7**	Sinusoidal Function Context and Data Modeling
2 3	**3.8**	The Tangent Function
1 2	**3.9**	Inverse Trigonometric Functions
1 2 3	**3.10**	Trigonometric Equations and Inequalities
2 3	**3.11**	The Secant, Cosecant, and Cotangent Functions
1 3	**3.12**	Equivalent Representations of Trigonometric Functions
1 2	**3.13**	Trigonometry and Polar Coordinates
2 3	**3.14**	Polar Function Graphs
3	**3.15**	Rates of Change in Polar Functions

UNIT 4 Functions Involving Parameters, Vectors, and Matrices

7 weeks | **0%** AP Exam Weighting

1 2	**4.1**	Parametric Functions
3	**4.2**	Parametric Functions Modeling Planar Motion
3	**4.3**	Parametric Functions and Rates of Change
1	**4.4**	Parametrically Defined Circles and Lines
2 3	**4.5**	Implicitly Defined Functions
1 2	**4.6**	Conic Sections
1 2	**4.7**	Parametrization of Implicitly Defined Functions
2 3	**4.8**	Vectors
3	**4.9**	Vector-Valued Functions
1 3	**4.10**	Matrices
1 3	**4.11**	The Inverse and Determinant of a Matrix
1	**4.12**	Linear Transformations and Matrices
1 2 3	**4.13**	Matrices as Functions
1 3	**4.14**	Matrices Modeling Contexts

Progress Check Unit 3 Part 1: Topics 3.1–3.7

Multiple-choice: 21
Free-response: 2

Progress Check Unit 3 Part 2: Topics 3.8–3.15

Multiple-choice: 24
Free-response: 2

Progress Check Unit 4 Part 1: Topics 4.1–4.7

Multiple-choice: 24
Free-response: 2

Progress Check Unit 4 Part 2: Topics 4.8–4.14

Multiple-choice: 21
Free-response: 2

AP PRECALCULUS

UNIT 1

Polynomial and Rational Functions

AP Precalculus Exam Topics
(required for college calculus placement)

30–40%
AP EXAM WEIGHTING

30–40
CLASS PERIODS

Remember to go to **AP Classroom** to assign students the online **Progress Checks** for this unit.

Whether assigned as homework or completed in class, the **Progress Checks** provide each student with immediate feedback related to this unit's topics and skills.

Progress Check Unit 1 Part 1: Topics 1.1–1.6

Multiple-choice: 18
Free-response: 2

Progress Check Unit 1 Part 2: Topics 1.7–1.14

Multiple-choice: 24
Free-response: 2

UNIT 1

30–40% AP EXAM WEIGHTING

30–40 CLASS PERIODS

Polynomial and Rational Functions

ESSENTIAL QUESTIONS

- How do we model the intensity of light from its source?
- How can I use data and graphs to figure out the best time to purchase event tickets?
- How can we adjust known projectile motion models to account for changes in conditions?

Developing Understanding

In Unit 1, students develop understanding of two key function concepts while exploring polynomial and rational functions. The first concept is covariation, or how output values change in tandem with changing input values. The second concept is rates of change, including average rate of change, rate of change at a point, and changing rates of change. The central idea of a function as a rule for relating two simultaneously changing sets of values provides students with a vital tool that has many applications, in nature, human society, and business and industry. For example, the idea of crop yield increasing but at a decreasing rate or the efficacy of a medicine decreasing but at an increasing rate are important understandings that inform critical decisions.

Building the Mathematical Practices

3.A **3.B**

Throughout the course, students should practice communicating mathematics and developing notational fluency—and that practice should begin in Unit 1. Students should use precise language such as, "On the closed interval 0 to 1, as the value of *x* increases, the value of *y* increases then decreases." To the fullest extent possible, students should work on functions presented in contextual scenarios such as graphs showing distance vs. time, tables showing velocity vs. time, or scenarios involving volume vs. time. In these contexts, students should use clear language when referring to variables and functions, including units of measure as appropriate. For example, when considering a problem of filling a pool with water, a student may write, "The input values of the function *V* are times in minutes, and the output values are volumes in cubic meters. The average rate of change of the function *V* over the time interval *t* equals 2 minutes to *t* equals 5 minutes is 0.4 cubic meters per minute." Practicing communicating with precise language can help students clarify their thinking and make important connections while revealing misconceptions.

Preparing for the AP Exam

After studying Unit 1, students should be able to describe, represent, and model polynomial and rational functions and their additive and multiplicative transformations. Because part of the exam relies on technology, students should be able to identify zeros, points of intersection, and extrema using graphing calculator technology. Students should be able to calculate linear, quadratic, cubic, and quartic regressions to model a data set. In the free-response section of the exam, students will not only be required to arrive at a solution but also explain and provide rationales for their conclusions. Students should practice providing reasons for conclusions throughout the unit in both spoken and written form and continually refine their explanations to improve precision.

UNIT AT A GLANCE

Topic	Instructional Periods	Suggested Skill Focus
1.1 Change in Tandem	2	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.2 Rates of Change	2	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.3 Rates of Change in Linear and Quadratic Functions	2	**3.B** Apply numerical results in a given mathematical or applied context. **3.C** Support conclusions or choices with a logical rationale or appropriate data.
1.4 Polynomial Functions and Rates of Change	2	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.5 Polynomial Functions and Complex Zeros	2–3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
1.6 Polynomial Functions and End Behavior	1–2	**3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.7 Rational Functions and End Behavior	2–3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.8 Rational Functions and Zeros	1–2	**1.A** Solve equations and inequalities represented analytically, with and without technology.

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UNIT AT A GLANCE *(cont'd)*

Topic	Instructional Periods	Suggested Skill Focus
1.9 **Rational Functions and Vertical Asymptotes**	1–2	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.
1.10 **Rational Functions and Holes**	1–2	**3.C** Support conclusions or choices with a logical rationale or appropriate data.
1.11 **Equivalent Representations of Polynomial and Rational Expressions**	2–3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.B** Apply numerical results in a given mathematical or applied context.
1.12 **Transformations of Functions**	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
1.13 **Function Model Selection and Assumption Articulation**	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.C** Support conclusions or choices with a logical rationale or appropriate data.
1.14 **Function Model Construction and Application**	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.

Go to **AP Classroom** to assign the **Progress Checks** for Unit 1.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 125 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	1.1	Students are given a list of variables such as time, temperature, speed, cost, intensity, length, height, volume, and air flow. The teacher sketches the graph of a function curve with unlabeled axes. In pairs, students develop a situation and story involving two of the variables that can be modeled by the given curve.
2	1.3 1.4	Students are given phrases such as "the function is increasing with a decreasing rate of change," "the function has an average rate of change of -4 on the interval $[2, 5]$," "the rates of change of a function are constant," and "the polynomial function is even and has a local minimum at $x = 2$." Students construct a graph that would be consistent with each phrase and compare their results.
3	1.7	Each student is given cards containing different rational functions in analytical representations. Have students use a calculator to graph the function and then record the intercepts on the card as well as limit expressions to describe the function's end behavior and behavior at each vertical or horizontal asymptote (e.g., $\lim\limits_{x\to3^+} f(x) = -\infty$, $\lim\limits_{x\to\infty} f(x) = 2$). In pairs, students take turns reading their limit statements to each other. Without seeing the actual rational function and without using a calculator, students will try to sketch the function's graph and then check and discuss. Have students rotate to form new pairs and repeat.
4	1.11	Students are presented with a nonconstant polynomial or rational function in analytical representations, and they then translate the expression into a variety of representations: constructing a graph, writing the expression as a product of linear factors $(x - a)$ when possible, and verbally describing characteristics such as real zeros, x-intercepts, asymptotes, and holes. Then have students check their graphs using technology.
5	1.11	Students are given statements and need to classify each statement as whether the statement is always true, sometimes true, or never true. They should justify or explain their choices. Sample statements include: "The graphs of all rational functions have a horizontal asymptote," "If $f(1) < 0$ and $f(3) > 0$, the polynomial function f must have a zero between $x = 1$ and $x = 3$," and "The graphs of rational functions have holes and vertical asymptotes."
6	1.12	Students are given graphs of polynomial and rational functions. Students are then asked to graph a transformation of one of the provided graphs, such as a vertical dilation by a factor of 3 and a horizontal translation of 2 units. Students will then switch with a peer and try to write the new expression for the function transformation. Students then have time to discuss the new function expressions and adjust as needed.

TOPIC 1.1

Change in Tandem

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.1.A Describe how the input and output values of a function vary together by comparing function values.	**1.1.A.1** A *function* is a mathematical relation that maps a set of input values to a set of output values such that each input value is mapped to exactly one output value. The set of input values is called the *domain* of the function, and the set of output values is called the *range* of the function. The variable representing input values is called the *independent variable*, and the variable representing output values is called the *dependent variable*. **1.1.A.2** The input and output values of a function vary in tandem according to the *function rule*, which can be expressed graphically, numerically, analytically, or verbally. **1.1.A.3** A function is increasing over an interval of its domain if, as the input values increase, the output values always increase. That is, for all a and b in the interval, if $a < b$, then $f(a) < f(b)$. **1.1.A.4** A function is decreasing over an interval of its domain if, as the input values increase, the output values always decrease. That is, for all a and b in the interval, if $a < b$, then $f(a) > f(b)$.

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Polynomial and Rational Functions

LEARNING OBJECTIVE

1.1.B

Construct a graph representing two quantities that vary with respect to each other in a contextual scenario.

ESSENTIAL KNOWLEDGE

1.1.B.1

The graph of a function displays a set of input-output pairs and shows how the values of the function's input and output values vary.

1.1.B.2

A verbal description of the way aspects of phenomena change together can be the basis for constructing a graph.

1.1.B.3

The graph of a function is *concave up* on intervals in which the rate of change is increasing.

1.1.B.4

The graph of a function is *concave down* on intervals in which the rate of change is decreasing.

1.1.B.5

The graph intersects the x-axis when the output value is zero. The corresponding input values are said to be *zeros of the function*.

TOPIC 1.2

Rates of Change

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.2.A Compare the rates of change at two points using average rates of change near the points.	**1.2.A.1** The average rate of change of a function over an interval of the function's domain is the constant rate of change that yields the same change in the output values as the function yielded on that interval of the function's domain. It is the ratio of the change in the output values to the change in input values over that interval. **1.2.A.2** The rate of change of a function at a point quantifies the rate at which output values would change were the input values to change at that point. The rate of change at a point can be approximated by the average rates of change of the function over small intervals containing the point, if such values exist. **1.2.A.3** The rates of change at two points can be compared using average rate of change approximations over sufficiently small intervals containing each point, if such values exist.
1.2.B Describe how two quantities vary together at different points and over different intervals of a function.	**1.2.B.1** Rates of change quantify how two quantities vary together. **1.2.B.2** A positive rate of change indicates that as one quantity increases or decreases, the other quantity does the same. **1.2.B.3** A negative rate of change indicates that as one quantity increases, the other decreases.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

3.B
Apply numerical results in a given mathematical or applied context.

3.C
Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 1.3

Rates of Change in Linear and Quadratic Functions

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.3.A Determine the average rates of change for sequences and functions, including linear, quadratic, and other function types.	**1.3.A.1** For a linear function, the average rate of change over any length input-value interval is constant. **1.3.A.2** For a quadratic function, the average rates of change over consecutive equal-length input-value intervals can be given by a linear function. **1.3.A.3** The average rate of change over the closed interval $[a,\ b]$ is the slope of the secant line from the point $(a, f(a))$ to $(b, f(b))$.
1.3.B Determine the change in the average rates of change for linear, quadratic, and other function types.	**1.3.B.1** For a linear function, since the average rates of change over consecutive equal-length input-value intervals can be given by a constant function, these average rates of change for a linear function are changing at a rate of zero. **1.3.B.2** For a quadratic function, since the average rates of change over consecutive equal-length input-value intervals can be given by a linear function, these average rates of change for a quadratic function are changing at a constant rate. **1.3.B.3** When the average rate of change over equal-length input-value intervals is increasing for all small-length intervals, the graph of the function is concave up. When the average rate of change over equal-length input-value intervals is decreasing for all small-length intervals, the graph of the function is concave down.

TOPIC 1.4

Polynomial Functions and Rates of Change

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

1.4.A

Identify key characteristics of polynomial functions related to rates of change.

ESSENTIAL KNOWLEDGE

1.4.A.1

A nonconstant polynomial function of x is any function representation that is equivalent to the analytical form $p(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$, where n is a positive integer, a_i is a real number for each i from 1 to n, and a_n is nonzero. The polynomial has degree n, the leading term is a_nx^n, and the leading coefficient is a_n. A constant is also a polynomial function of degree zero.

1.4.A.2

Where a polynomial function switches between increasing and decreasing, or at the included endpoint of a polynomial with a restricted domain, the polynomial function will have a *local*, or *relative*, maximum or minimum output value. Of all local maxima, the greatest is called the *global*, or *absolute*, maximum. Likewise, the least of all local minima is called the global, or absolute, minimum.

1.4.A.3

Between every two distinct real zeros of a nonconstant polynomial function, there must be at least one input value corresponding to a local maximum or local minimum.

1.4.A.4

Polynomial functions of an even degree will have either a global maximum or a global minimum.

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LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.4.A Identify key characteristics of polynomial functions related to rates of change.	**1.4.A.5** *Points of inflection* of a polynomial function occur at input values where the rate of change of the function changes from increasing to decreasing or from decreasing to increasing. This occurs where the graph of a polynomial function changes from concave up to concave down or from concave down to concave up.

TOPIC 1.5

Polynomial Functions and Complex Zeros

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

Required Course Content

LEARNING OBJECTIVE

1.5.A

Identify key characteristics of a polynomial function related to its zeros when suitable factorizations are available or with technology.

ESSENTIAL KNOWLEDGE

1.5.A.1

If a is a complex number and $p(a) = 0$, then a is called a *zero* of the polynomial function p, or a *root* of $p(x) = 0$. If a is a real number, then $(x - a)$ is a linear factor of p if and only if a is a zero of p.

1.5.A.2

If a linear factor $(x - a)$ is repeated n times, the corresponding zero of the polynomial function has a *multiplicity* n. A polynomial function of *degree* n has exactly n complex zeros when counting multiplicities.

1.5.A.3

If a is a real zero of a polynomial function p, then the graph of $y = p(x)$ has an x-intercept at the point $(a, 0)$. Consequently, real zeros of a polynomial can be endpoints for intervals satisfying polynomial inequalities.

1.5.A.4

If $a + bi$ is a non-real zero of a polynomial function p, then its *conjugate* $a - bi$ is also a zero of p.

1.5.A.5

If the real zero, a, of a polynomial function has even multiplicity, then the signs of the output values are the same for input values near $x = a$. For these polynomial functions, the graph will be tangent to the x-axis at $x = a$.

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LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.5.A Identify key characteristics of a polynomial function related to its zeros when suitable factorizations are available or with technology.	**1.5.A.6** The degree of a polynomial function can be found by examining the successive differences of the output values over equal-interval input values. The degree of the polynomial function is equal to the least value n for which the successive $n\text{th}$ differences are constant.
1.5.B Determine if a polynomial function is even or odd.	**1.5.B.1** An *even* function is graphically symmetric over the line $x = 0$ and analytically has the property $f(-x) = f(x)$. If n is even, then a polynomial of the form $p(x) = a_n x^n$, where $n \geq 1$ and $a_n \neq 0$, is an even function. **1.5.B.2** An *odd* function is graphically symmetric about the point $(0,0)$ and analytically has the property $f(-x) = -f(x)$. If n is odd, then a polynomial of the form $p(x) = a_n x^n$, where $n \geq 1$ and $a_n \neq 0$, is an odd function.

TOPIC 1.6

Polynomial Functions and End Behavior

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.6.A Describe end behaviors of polynomial functions.	**1.6.A.1** As input values of a nonconstant polynomial function increase without bound, the output values will either increase or decrease without bound. The corresponding mathematical notation is $\lim\limits_{x \to \infty} p(x) = \infty$ or $\lim\limits_{x \to \infty} p(x) = -\infty$. **1.6.A.2** As input values of a nonconstant polynomial function decrease without bound, the output values will either increase or decrease without bound. The corresponding mathematical notation is $\lim\limits_{x \to -\infty} p(x) = \infty$ or $\lim\limits_{x \to -\infty} p(x) = -\infty$. **1.6.A.3** The degree and sign of the leading term of a polynomial determines the end behavior of the polynomial function, because as the input values increase or decrease without bound, the values of the leading term dominate the values of all lower-degree terms.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 1.7

Rational Functions and End Behavior

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.7.A Describe end behaviors of rational functions.	**1.7.A.1** A rational function is analytically represented as a quotient of two polynomial functions and gives a measure of the relative size of the polynomial function in the numerator compared to the polynomial function in the denominator for each value in the rational function's domain. **1.7.A.2** The end behavior of a rational function will be affected most by the polynomial with the greater degree, as its values will dominate the values of the rational function for input values of large magnitude. For input values of large magnitude, a polynomial is dominated by its leading term. Therefore, the end behavior of a rational function can be understood by examining the corresponding quotient of the leading terms. **1.7.A.3** If the polynomial in the numerator dominates the polynomial in the denominator for input values of large magnitude, then the quotient of the leading terms is a nonconstant polynomial, and the original rational function has the end behavior of that polynomial. If that polynomial is linear, then the graph of the rational function has a slant asymptote parallel to the graph of the line.

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LEARNING OBJECTIVE

1.7.A

Describe end behaviors of rational functions.

ESSENTIAL KNOWLEDGE

1.7.A.4

If neither polynomial in a rational function dominates the other for input values of large magnitude, then the quotient of the leading terms is a constant, and that constant indicates the location of a horizontal asymptote of the graph of the original rational function.

1.7.A.5

If the polynomial in the denominator dominates the polynomial in the numerator for input values of large magnitude, then the quotient of the leading terms is a rational function with a constant in the numerator and nonconstant polynomial in the denominator, and the graph of the original rational function has a horizontal asymptote at $y = 0$.

1.7.A.6

When the graph of a rational function r has a horizontal asymptote at $y = b$, where b is a constant, the output values of the rational function get arbitrarily close to b and stay arbitrarily close to b as input values increase or decrease without bound. The corresponding mathematical notation is $\lim_{x \to \infty} r(x) = b$ or $\lim_{x \to -\infty} r(x) = b$.

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

1.A

Solve equations and inequalities represented analytically, with and without technology.

TOPIC 1.8

Rational Functions and Zeros

Required Course Content

LEARNING OBJECTIVE

1.8.A

Determine the zeros of rational functions.

ESSENTIAL KNOWLEDGE

1.8.A.1

The real zeros of a rational function correspond to the real zeros of the numerator for such values in its domain.

1.8.A.2

The real zeros of both polynomial functions of a rational function r are endpoints or asymptotes for intervals satisfying the rational function inequalities $r(x) \geq 0$ or $r(x) \leq 0$.

TOPIC 1.9

Rational Functions and Vertical Asymptotes

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

Required Course Content

LEARNING OBJECTIVE

1.9.A

Determine vertical asymptotes of graphs of rational functions.

ESSENTIAL KNOWLEDGE

1.9.A.1

If the value a is a real zero of the polynomial function in the denominator of a rational function and is not also a real zero of the polynomial function in the numerator, then the graph of the rational function has a vertical asymptote at $x = a$. Furthermore, a vertical asymptote also occurs at $x = a$ if the multiplicity of a as a real zero in the denominator is greater than its multiplicity as a real zero in the numerator.

1.9.A.2

Near a vertical asymptote, $x = a$, of a rational function, the values of the polynomial function in the denominator are arbitrarily close to zero, so the values of the rational function r increase or decrease without bound. The corresponding mathematical notation is $\lim_{x \to a^+} r(x) = \infty$ or $\lim_{x \to a^+} r(x) = -\infty$ for input values near a and greater than a, and $\lim_{x \to a^-} r(x) = \infty$ or $\lim_{x \to a^-} r(x) = -\infty$ for input values near a and less than a.

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

3.C

Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 1.10

Rational Functions and Holes

Required Course Content

LEARNING OBJECTIVE

1.10.A

Determine holes in graphs of rational functions.

ESSENTIAL KNOWLEDGE

1.10.A.1

If the multiplicity of a real zero in the numerator is greater than or equal to its multiplicity in the denominator, then the graph of the rational function has a hole at the corresponding input value.

1.10.A.2

If the graph of a rational function r has a hole at $x = c$, then the location of the hole can be determined by examining the output values corresponding to input values sufficiently close to c. If input values sufficiently close to c correspond to output values arbitrarily close to L, then the hole is located at the point with coordinates (c, L). The corresponding mathematical notation is $\lim_{x \to c} r(x) = L$. It should be noted that

$$\lim_{x \to c^-} r(x) = \lim_{x \to c^+} r(x) = \lim_{x \to c} r(x) = L.$$

TOPIC 1.11

Equivalent Representations of Polynomial and Rational Expressions

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.B
Apply numerical results in a given mathematical or applied context.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.11.A Rewrite polynomial and rational expressions in equivalent forms.	**1.11.A.1** Because the factored form of a polynomial or rational function readily provides information about real zeros, it can reveal information about x-intercepts, asymptotes, holes, domain, and range. **1.11.A.2** The standard form of a polynomial or rational function can reveal information about end behaviors of the function. **1.11.A.3** The information extracted from different analytic representations of the same polynomial or rational function can be used to answer questions in context.
1.11.B Determine the quotient of two polynomial functions using long division.	**1.11.B.1** Polynomial long division is an algebraic process similar to numerical long division involving a quotient and remainder. If the polynomial f is divided by the polynomial g, then f can be rewritten as $f(x) = g(x)q(x) + r(x)$, where q is the quotient, r is the remainder, and the degree of r is less than the degree of g. **1.11.B.2** The result of polynomial long division is helpful in finding equations of slant asymptotes for graphs of rational functions.

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LEARNING OBJECTIVE

1.11.C

Rewrite the repeated product of binomials using the binomial theorem.

ESSENTIAL KNOWLEDGE

1.11.C.1

The binomial theorem utilizes the entries in a single row of Pascal's Triangle to more easily expand expressions of the form $(a+b)^n$, including polynomial functions of the form $p(x)=(x+c)^n$, where c is a constant.

TOPIC 1.12

Transformations of Functions

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

1.12.A

Construct a function that is an additive and/or multiplicative transformation of another function.

ESSENTIAL KNOWLEDGE

1.12.A.1

The function $g(x) = f(x) + k$ is an additive transformation of the function f that results in a vertical translation of the graph of f by k units.

1.12.A.2

The function $g(x) = f(x + h)$ is an additive transformation of the function f that results in a horizontal translation of the graph of f by $-h$ units.

1.12.A.3

The function $g(x) = af(x)$, where $a \neq 0$, is a multiplicative transformation of the function f that results in a vertical dilation of the graph of f by a factor of $|a|$. If $a < 0$, the transformation involves a reflection over the x-axis.

1.12.A.4

The function $g(x) = f(bx)$, where $b \neq 0$, is a multiplicative transformation of the function f that results in a horizontal dilation of the graph of f by a factor of $\left|\frac{1}{b}\right|$. If $b < 0$, the transformation involves a reflection over the y-axis.

1.12.A.5

Additive and multiplicative transformations can be combined, resulting in combinations of horizontal and vertical translations and dilations.

1.12.A.6

The domain and range of a function that is a transformation of a parent function may be different from those of the parent function.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.C
Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 1.13

Function Model Selection and Assumption Articulation

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.13.A Identify an appropriate function type to construct a function model for a given scenario.	**1.13.A.1** Linear functions model data sets or aspects of contextual scenarios that demonstrate roughly constant rates of change. **1.13.A.2** Quadratic functions model data sets or aspects of contextual scenarios that demonstrate roughly linear rates of change, or data sets that are roughly symmetric with a unique maximum or minimum value. **1.13.A.3** Geometric contexts involving area or two dimensions can often be modeled by quadratic functions. Geometric contexts involving volume or three dimensions can often be modeled by cubic functions. **1.13.A.4** Polynomial functions model data sets or contextual scenarios with multiple real zeros or multiple maxima or minima. **1.13.A.5** A polynomial function of degree n models data sets or contextual scenarios that demonstrate roughly constant nonzero $n\text{th}$ differences. **1.13.A.6** A polynomial function of degree n or less can be used to model a graph of $n+1$ points with distinct input values.

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LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.13.A Identify an appropriate function type to construct a function model for a given scenario.	**1.13.A.7** A piecewise-defined function consists of a set of functions defined over nonoverlapping domain intervals and is useful for modeling a data set or contextual scenario that demonstrates different characteristics over different intervals.
1.13.B Describe assumptions and restrictions related to building a function model.	**1.13.B.1** A model may have underlying assumptions about what is consistent in the model. **1.13.B.2** A model may have underlying assumptions about how quantities change together. **1.13.B.3** A model may require domain restrictions based on mathematical clues, contextual clues, or extreme values in the data set. **1.13.B.4** A model may require range restrictions, such as rounding values, based on mathematical clues, contextual clues, or extreme values in the data set.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C
Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.B
Apply numerical results in a given mathematical or applied context.

TOPIC 1.14

Function Model Construction and Application

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
1.14.A Construct a linear, quadratic, cubic, quartic, polynomial of degree n, or related piecewise-defined function model.	**1.14.A.1** A model can be constructed based on restrictions identified in a mathematical or contextual scenario. **1.14.A.2** A model of a data set or a contextual scenario can be constructed using transformations of the parent function. **1.14.A.3** A model of a data set can be constructed using technology and regressions, including linear, quadratic, cubic, and quartic regressions. **1.14.A.4** A piecewise-defined function model can be constructed through a combination of modeling techniques.
1.14.B Construct a rational function model based on a context.	**1.14.B.1** Data sets and aspects of contextual scenarios involving quantities that are inversely proportional can often be modeled by rational functions. For example, the magnitudes of both gravitational force and electromagnetic force between objects are inversely proportional to the objects' squared distance.
1.14.C Apply a function model to answer questions about a data set or contextual scenario.	**1.14.C.1** A model can be used to draw conclusions about the modeled data set or contextual scenario, including answering key questions and predicting values, rates of change, average rates of change, and changing rates of change. Appropriate units of measure should be extracted or inferred from the given context.

AP PRECALCULUS

UNIT 2

Exponential and Logarithmic Functions

AP Precalculus Exam Topics
(required for college calculus placement)

27–40%
AP EXAM WEIGHTING

30–45
CLASS PERIODS

Remember to go to **AP Classroom** to assign students the online **Progress Checks** for this unit.

Whether assigned as homework or completed in class, the **Progress Checks** provide each student with immediate feedback related to this unit's topics and skills.

Progress Check Unit 2 Part 1: Topics 2.1–2.8

Multiple-choice: 24
Free-response: 2

Progress Check Unit 2 Part 2: Topics 2.9–2.15

Multiple-choice: 24
Free-response: 2

UNIT 2

27–40% AP EXAM WEIGHTING

30–45 CLASS PERIODS

Exponential and Logarithmic Functions

ESSENTIAL QUESTIONS

- How can I make a single model that merges the interest I earn from my bank with the taxes that are due so I can know how much I will have in the end?
- How can we adjust the scale of distance for a model of planets in the solar system so the relationships among the planets are easier to see?
- If different functions can be used to model data, how do we pick which one is best?

Developing Understanding

In Unit 2, students build an understanding of exponential and logarithmic functions. Exponential and logarithmic function models are widespread in the natural and social sciences. When an aspect of a phenomenon changes proportionally to the existing amount, exponential and logarithmic models are employed to harness the information. Exponential functions are key to modeling population growth, radioactive decay, interest rates, and the amount of medication in a patient. Logarithmic functions are useful in modeling sound intensity and frequency, the magnitude of earthquakes, the pH scale in chemistry, and the working memory in humans. The study of these two function types touches careers in business, medicine, chemistry, physics, education, and human geography, among others.

Building the Mathematical Practices

2.A 2.B

Students should learn to communicate differences and similarities among arithmetic sequences, linear functions, geometric sequences, and exponential functions. Students can develop a deeper understanding of these four function types by considering how each would be represented in a graph, in a table, in an analytical representation, and through verbal descriptions of related scenarios. Examining multiple representations is also powerful in understanding composition of functions and relationships between functions and their inverse functions. In this unit, multiple representations should be used to explore the inverse relationship between exponential and logarithmic functions.

Preparing for the AP Exam

Students should practice clearly delineating the processes that lead to answers when solving equations, finding equivalent expressions, and building function models. Since manipulating exponential and logarithmic functions requires a great deal of precision, students should avoid "skipping steps" as errors can be easily introduced with these function types. Furthermore, on the free-response section of the AP Exam, answers without supporting work may not be acceptable, so students should practice clear communication throughout the course. Students will continue to use technology to explore multiple representations in this unit as they did in Unit 1 and will expand their use of calculating regressions to include exponential and logarithmic regressions. They should practice calculating various regressions on data sets, plot the residuals, and justify the choice of a function model based on analysis of the residuals. Students should be fluent with the practices of entering data, calculating regressions, and plotting residuals with the graphing calculator before the AP Exam.

UNIT AT A GLANCE

Topic	Instructional Periods	Suggested Skill Focus
2.1 Change in Arithmetic and Geometric Sequences	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
2.2 Change in Linear and Exponential Functions	2	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.
2.3 Exponential Functions	1–2	**3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
2.4 Exponential Function Manipulation	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
2.5 Exponential Function Context and Data Modeling	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.
2.6 Competing Function Model Validation	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.C** Support conclusions or choices with a logical rationale or appropriate data.
2.7 Composition of Functions	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
2.8 Inverse Functions	2–3	**1.A** Solve equations and inequalities represented analytically, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

continued on next page

UNIT AT A GLANCE *(cont'd)*

Topic	Instructional Periods	Suggested Skill Focus
2.9 Logarithmic Expressions	1–2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.
2.10 Inverses of Exponential Functions	2	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
2.11 Logarithmic Functions	1–2	**3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
2.12 Logarithmic Function Manipulation	2–3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
2.13 Exponential and Logarithmic Equations and Inequalities	3–4	**1.A** Solve equations and inequalities represented analytically, with and without technology. **1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.
2.14 Logarithmic Function Context and Data Modeling	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.
2.15 Semi-log Plots	2–3	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.C** Support conclusions or choices with a logical rationale or appropriate data.

Go to **AP Classroom** to assign the **Progress Checks** for Unit 2.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 125 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	2.2	Five students randomly choose different integers between -6 and 6 that will be used as input values for a function. In groups, students determine appropriate output values for the five input values that illustrate a linear function relationship, then those that illustrate an exponential function relationship. The groups defend their decisions to the rest of the class by explaining how they know their values represent linear and exponential functions.
2	2.2	Have students pair up and attempt to walk (in a straight line from a specific starting point) using the following walking patterns: i) at a constant rate of change of 2 feet per second based on the model $d = 2t$; ii) at an increasing rate of change based on the model $d = 2^t$. Ask each pair to describe the differences in their two experiences. Follow up with applied questions such as "How far would you walk in 10 seconds? in 33 seconds?" or "Using each model, how long would it take you to travel around Earth (~25,000 miles)?, travel to the Moon (~239,000 miles)?"
3	2.5	Student *A* and Student *B* are both provided with a context that can be modeled using exponential functions such as doubling or halving a certain substance. Without looking at each other's work, each student generates a graph and an expression to model the exponential function. Then, the students switch papers to review each other's solutions.
4	2.6	Students are given multiple tables of data, each of which is well-modeled by a linear, exponential, or quadratic function. In pairs, students calculate all three regressions, observe the graphs of the regressions, observe the corresponding residual plots, and draw conclusions about the relationships between the graphs and the residuals.
5	2.11	Students are given a sheet of paper containing several logarithmic functions represented analytically and their corresponding graphs, in mixed order, in a separate column. In small groups, students match the corresponding representations and take turns explaining how they know each is matched correctly.
6	2.13	Place 10 to 16 index cards around the room. The top of each card contains a problem, and the bottom of each card contains a solution to a problem from a different card. Students will start at a card, solve the problem, and then search for the next card with their solution. The cards should be set up such that the students will find the matches of all of the cards.

TOPIC 2.1

Change in Arithmetic and Geometric Sequences

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.1.A Express arithmetic sequences found in mathematical and contextual scenarios as functions of the whole numbers.	**2.1.A.1** A *sequence* is a function from the whole numbers to the real numbers. Consequently, the graph of a sequence consists of discrete points instead of a curve. **2.1.A.2** Successive terms in an arithmetic sequence have a common difference, or constant rate of change. **2.1.A.3** The general term of an arithmetic sequence with a common difference d is denoted by a_n and is given by $a_n = a_0 + dn$, where a_0 is the initial value, or by $a_n = a_k + d(n - k)$, where a_k is the kth term of the sequence.
2.1.B Express geometric sequences found in mathematical and contextual scenarios as functions of the whole numbers.	**2.1.B.1** Successive terms in a geometric sequence have a common ratio, or constant proportional change. **2.1.B.2** The general term of a geometric sequence with a common ratio r is denoted by g_n and is given by $g_n = g_0 r^n$, where g_0 is the initial value, or by $g_n = g_k r^{(n-k)}$, where g_k is the kth term of the sequence. **2.1.B.3** Increasing arithmetic sequences increase equally with each step, whereas increasing geometric sequences increase by a larger amount with each successive step.

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INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.C
Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.B
Apply numerical results in a given mathematical or applied context.

TOPIC 2.2

Change in Linear and Exponential Functions

Required Course Content

LEARNING OBJECTIVE

2.2.A
Construct functions of the real numbers that are comparable to arithmetic and geometric sequences.

ESSENTIAL KNOWLEDGE

2.2.A.1
Linear functions of the form $f(x) = b + mx$ are similar to arithmetic sequences of the form $a_n = a_0 + dn$, as both can be expressed as an initial value (b or a_0) plus repeated addition of a constant rate of change, the slope (m or d).

2.2.A.2
Similar to arithmetic sequences of the form $a_n = a_k + d(n - k)$, which are based on a known difference, d, and a kth term, linear functions can be expressed in the form $f(x) = y_i + m(x - x_i)$ based on a known slope, m, and a point, (x_i, y_i).

2.2.A.3
Exponential functions of the form $f(x) = ab^x$ are similar to geometric sequences of the form $g_n = g_0 r^n$, as both can be expressed as an initial value (a or g_0) times repeated multiplication by a constant proportion (b or r).

2.2.A.4
Similar to geometric sequences of the form $g_n = g_k r^{(n-k)}$, which are based on a known ratio, r, and a kth term, exponential functions can be expressed in the form $f(x) = y_i r^{(x-x_i)}$ based on a known ratio, r, and a point, (x_i, y_i).

2.2.A.5
Sequences and their corresponding functions may have different domains.

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LEARNING OBJECTIVE

2.2.B

Describe similarities and differences between linear and exponential functions.

ESSENTIAL KNOWLEDGE

2.2.B.1

Over equal-length input-value intervals, if the output values of a function change at constant rate, then the function is linear; if the output values of a function change proportionally, then the function is exponential.

2.2.B.2

Linear functions of the form $f(x) = b + mx$ and exponential functions of the form $f(x) = ab^x$ can both be expressed analytically in terms of an initial value and a constant involved with change. However, linear functions are based on addition, while exponential functions are based on multiplication.

2.2.B.3

Arithmetic sequences, linear functions, geometric sequences, and exponential functions all have the property that they can be determined by two distinct sequence or function values.

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 2.3

Exponential Functions

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.3.A Identify key characteristics of exponential functions.	**2.3.A.1** The general form of an exponential function is $f(x) = ab^x$, with the *initial value* a, where $a \neq 0$, and the *base* b, where $b > 0$, and $b \neq 1$. When $a > 0$ and $b > 1$, the exponential function is said to demonstrate *exponential growth*. When $a > 0$ and $0 < b < 1$, the exponential function is said to demonstrate *exponential decay*. **2.3.A.2** When the natural numbers are input values in an exponential function, the input value specifies the number of factors of the base to be applied to the function's initial value. The domain of an exponential function is all real numbers. **2.3.A.3** Because the output values of exponential functions in general form are proportional over equal-length input-value intervals, exponential functions are always increasing or always decreasing, and their graphs are always concave up or always concave down. Consequently, exponential functions do not have extrema except on a closed interval, and their graphs do not have points of inflection. **2.3.A.4** If the values of the additive transformation function $g(x) = f(x) + k$ of any function f are proportional over equal-length input-value intervals, then f is exponential.

LEARNING OBJECTIVE

2.3.A

Identify key characteristics of exponential functions.

ESSENTIAL KNOWLEDGE

2.3.A.5

For an exponential function in general form, as the input values increase or decrease without bound, the output values will increase or decrease without bound or will get arbitrarily close to zero. That is, for an exponential function in general form, $\lim\limits_{x \to \pm\infty} ab^x = \infty$, $\lim\limits_{x \to \pm\infty} ab^x = -\infty$, or $\lim\limits_{x \to \pm\infty} ab^x = 0$.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 2.4

Exponential Function Manipulation

Required Course Content

LEARNING OBJECTIVE

2.4.A
Rewrite exponential expressions in equivalent forms.

ESSENTIAL KNOWLEDGE

2.4.A.1
The product property for exponents states that $b^m b^n = b^{(m+n)}$. Graphically, this property implies that every horizontal translation of an exponential function, $f(x) = b^{(x+k)}$, is equivalent to a vertical dilation, $f(x) = b^{(x+k)} = b^x b^k = ab^x$, where $a = b^k$.

2.4.A.2
The power property for exponents states that $\left(b^m\right)^n = b^{(mn)}$. Graphically, this property implies that every horizontal dilation of an exponential function, $f(x) = b^{(cx)}$, is equivalent to a change of the base of an exponential function, $f(x) = \left(b^c\right)^x$, where b^c is a constant and $c \neq 0$.

2.4.A.3
The negative exponent property states that $b^{-n} = \frac{1}{b^n}$.

2.4.A.4
The value of an exponential expression involving an exponential unit fraction, such as $b^{(1/k)}$ where k is a natural number, is the kth root of b, when it exists.

TOPIC 2.5

Exponential Function Context and Data Modeling

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.B

Apply numerical results in a given mathematical or applied context.

Required Course Content

LEARNING OBJECTIVE

2.5.A

Construct a model for situations involving proportional output values over equal-length input-value intervals.

ESSENTIAL KNOWLEDGE

2.5.A.1

Exponential functions model growth patterns where successive output values over equal-length input-value intervals are proportional. When the input values are whole numbers, exponential functions model situations of repeated multiplication of a constant to an initial value.

2.5.A.2

A constant may need to be added to the dependent variable values of a data set to reveal a proportional growth pattern.

2.5.A.3

An exponential function model can be constructed from an appropriate ratio and initial value or from two input-output pairs. The initial value and the base can be found by solving a system of equations resulting from the two input-output pairs.

2.5.A.4

Exponential function models can be constructed by applying transformations to $f(x) = ab^x$ based on characteristics of a contextual scenario or data set.

2.5.A.5

Exponential function models can be constructed for a data set with technology using exponential regressions.

2.5.A.6

The natural base e, which is approximately 2.718, is often used as the base in exponential functions that model contextual scenarios.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.5.B Apply exponential models to answer questions about a data set or contextual scenario.	**2.5.B.1** For an exponential model in general form $f(x) = ab^x$, the base of the exponent, b, can be understood as a growth factor in successive unit changes in the input values and is related to a percent change in context. **2.5.B.2** Equivalent forms of an exponential function can reveal different properties of the function. For example, if d represents number of days, then the base of $f(d) = 2^d$ indicates that the quantity increases by a factor of 2 every day, but the equivalent form $f(d) = \left(2^7\right)^{(d/7)}$ indicates that the quantity increases by a factor of 2^7 every week. **2.5.B.3** Exponential models can be used to predict values for the dependent variable, depending on the contextual constraints on the domain.

TOPIC 2.6

Competing Function Model Validation

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.C
Support conclusions or choices with a logical rationale or appropriate data.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.6.A Construct linear, quadratic, and exponential models based on a data set.	**2.6.A.1** Two variables in a data set that demonstrate a slightly changing rate of change can be modeled by linear, quadratic, and exponential function models. **2.6.A.2** Models can be compared based on contextual clues and applicability to determine which model is most appropriate.
2.6.B Validate a model constructed from a data set.	**2.6.B.1** A model is justified as *appropriate* for a data set if the graph of the residuals of a regression, the residual plot, appear without pattern. **2.6.B.2** The difference between the predicted and actual values is the *error* in the model. Depending on the data set and context, it may be more appropriate to have an underestimate or overestimate for any given interval.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

TOPIC 2.7

Composition of Functions

Required Course Content

LEARNING OBJECTIVE

2.7.A

Evaluate the composition of two or more functions for given values.

ESSENTIAL KNOWLEDGE

2.7.A.1

If f and g are functions, the composite function $f \circ g$ maps a set of input values to a set of output values such that the output values of g are used as input values of f. For this reason, the domain of the composite function is restricted to those input values of g for which the corresponding output value is in the domain of f. $(f \circ g)(x)$ can also be represented as $f(g(x))$.

2.7.A.2

Values for the composite function $f \circ g$ can be calculated or estimated from the graphical, numerical, analytical, or verbal representations of f and g by using output values from g as input values for f.

2.7.A.3

The composition of functions is not commutative; that is, $f \circ g$ and $g \circ f$ are typically different functions; therefore, $f(g(x))$ and $g(f(x))$ are typically different values.

2.7.A.4

If the function $f(x) = x$ is composed with any function g, the resulting composite function is the same as g; that is, $g(f(x)) = f(g(x)) = g(x)$. The function $f(x) = x$ is called the *identity function*. When composing two functions, the identify function acts in the same way as 0, the additive identity, when adding two numbers and 1, the multiplicative identity, when multiplying two numbers.

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LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.7.B Construct a representation of the composition of two or more functions.	**2.7.B.1** Function composition is useful for relating two quantities that are not directly related by an existing formula.
	2.7.B.2 When analytic representations of the functions f and g are available, an analytic representation of $f(g(x))$ can be constructed by substituting $g(x)$ for every instance of x in f.
	2.7.B.3 A numerical or graphical representation of $f \circ g$ can often be constructed by calculating or estimating values for $(x, f(g(x)))$.
2.7.C Rewrite a given function as a composition of two or more functions.	**2.7.C.1** Functions given analytically can often be decomposed into less complicated functions. When properly decomposed, the variable in one function should replace each instance of the function with which it was composed.
	2.7.C.2 An additive transformation of a function, f, that results in vertical and horizontal translations can be understood as the composition of $g(x) = x + k$ with f.
	2.7.C.3 A multiplicative transformation of a function, f, that results in vertical and horizontal dilations can be understood as the composition of $g(x) = kx$ with f.

TOPIC 2.8

Inverse Functions

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.A
Solve equations and inequalities represented analytically, with and without technology.

2.B
Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.8.A Determine the input-output pairs of the inverse of a function.	**2.8.A.1** On a specified domain, a function, f, has an inverse function, or is invertible, if each output value of f is mapped from a unique input value. The domain of a function may be restricted in many ways to make the function invertible. **2.8.A.2** An inverse function can be thought of as a reverse mapping of the function. An inverse function, f^{-1}, maps the output values of a function, f, on its invertible domain to their corresponding input values; that is, if $f(a) = b$, then $f^{-1}(b) = a$. Alternately, on its invertible domain, if a function consists of input-output pairs (a, b), then the inverse function consists of input-output pairs (b, a).
2.8.B Determine the inverse of a function on an invertible domain.	**2.8.B.1** The composition of a function, f, and its inverse function, f^{-1}, is the identity function; that is, $f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x$. **2.8.B.2** On a function's invertible domain, the function's range and domain are the inverse function's domain and range, respectively. The inverse of the table of values of $y = f(x)$ can be found by reversing the input-output pairs; that is, (a, b) corresponds to (b, a).

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LEARNING OBJECTIVE

2.8.B

Determine the inverse of a function on an invertible domain.

ESSENTIAL KNOWLEDGE

2.8.B.3

The inverse of the graph of the function $y = f(x)$ can be found by reversing the roles of the x- and y-axes; that is, by reflecting the graph of the function over the graph of the identity function $h(x) = x$.

2.8.B.4

The inverse of the function can be found by determining the inverse operations to reverse the mapping. One method for finding the inverse of the function f is reversing the roles of x and y in the equation $y = f(x)$, then solving for $y = f^{-1}(x)$.

2.8.B.5

In addition to limiting the domain of a function to obtain an inverse function, contextual restrictions may also limit the applicability of an inverse function.

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

TOPIC 2.9

Logarithmic Expressions

Required Course Content

LEARNING OBJECTIVE

2.9.A
Evaluate logarithmic expressions.

ESSENTIAL KNOWLEDGE

2.9.A.1
The logarithmic expression $\log_b c$ is equal to, or represents, the value that the base b must be exponentially raised to in order to obtain the value c. That is, $\log_b c = a$ if and only if $b^a = c$, where a and c are constants, $b > 0$, and $b \neq 1$. (when the base of a logarithmic expression is not specified, it is understood as the common logarithm with a base of 10)

2.9.A.2
The values of some logarithmic expressions are readily accessible through basic arithmetic while other values can be estimated through the use of technology.

2.9.A.3
On a logarithmic scale, each unit represents a multiplicative change of the base of the logarithm. For example, on a standard scale, the units might be $0, 1, 2, \ldots$, while on a logarithmic scale, using logarithm base 10, the units might be $10^0, 10^1, 10^2, \ldots$.

TOPIC 2.10

Inverses of Exponential Functions

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

Required Course Content

LEARNING OBJECTIVE

2.10.A

Construct representations of the inverse of an exponential function with an initial value of 1.

ESSENTIAL KNOWLEDGE

2.10.A.1

The general form of a logarithmic function is $f(x) = a\log_b x$, with *base* b, where $b > 0, b \neq 1$, and $a \neq 0$.

2.10.A.2

The way in which input and output values vary together have an inverse relationship in exponential and logarithmic functions. Output values of general-form exponential functions change proportionately as input values increase in equal-length intervals. However, input values of general-form logarithmic functions change proportionately as output values increase in equal-length intervals. Alternately, exponential growth is characterized by output values changing multiplicatively as input values change additively, whereas logarithmic growth is characterized by output values changing additively as input values change multiplicatively.

2.10.A.3

$f(x) = \log_b x$ and $g(x) = b^x$, where $b > 0$ and $b \neq 1$, are inverse functions. That is, $g(f(x)) = f(g(x)) = x$.

2.10.A.4

The graph of the logarithmic function $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$, is a reflection of the graph of the exponential function $g(x) = b^x$, where $b > 0$ and $b \neq 1$, over the graph of the identity function $h(x) = x$.

2.10.A.5

If (s, t) is an ordered pair of the exponential function $g(x) = b^x$, where $b > 0$ and $b \neq 1$, then (t, s) is an ordered pair of the logarithmic function $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.

INSTRUCTIONAL PERIODS: 1–2

SKILLS FOCUS

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 2.11

Logarithmic Functions

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.11.A Identify key characteristics of logarithmic functions.	**2.11.A.1** The domain of a logarithmic function in general form is any real number greater than zero, and its range is all real numbers. **2.11.A.2** Because logarithmic functions are inverses of exponential functions, logarithmic functions are also always increasing or always decreasing, and their graphs are either always concave up or always concave down. Consequently, logarithmic functions do not have extrema except on a closed interval, and their graphs do not have points of inflection. **2.11.A.3** The additive transformation function $g(x) = f(x + k)$, where $k \neq 0$, of a logarithmic function f in general form does not have input values that are proportional over equal-length output-value intervals. However, if the input values of the additive transformation function, $g(x) = f(x + k)$, of any function f are proportional over equal-length output value intervals, then f is logarithmic. **2.11.A.4** With their limited domain, logarithmic functions in general form are vertically asymptotic to $x = 0$, with an end behavior that is unbounded. That is, for a logarithmic function in general form, $\lim\limits_{x \to 0^+} a \log_b x = \pm\infty$ and $\lim\limits_{x \to \infty} a \log_b x = \pm\infty$.

TOPIC 2.12

Logarithmic Function Manipulation

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

2.12.A

Rewrite logarithmic expressions in equivalent forms.

ESSENTIAL KNOWLEDGE

2.12.A.1

The product property for logarithms states that $\log_b(xy) = \log_b x + \log_b y$. Graphically, this property implies that every horizontal dilation of a logarithmic function, $f(x) = \log_b(kx)$, is equivalent to a vertical translation, $f(x) = \log_b(kx) = \log_b k + \log_b x = a + \log_b x$, where $a = \log_b k$.

2.12.A.2

The power property for logarithms states that $\log_b x^n = n\log_b x$. Graphically, this property implies that raising the input of a logarithmic function to a power, $f(x) = \log_b x^k$, results in a vertical dilation, $f(x) = \log_b x^k = k\log_b x$.

2.12.A.3

The change of base property for logarithms states that $\log_b x = \frac{\log_a x}{\log_a b}$, where $a > 0$ and $a \neq 1$. This implies that all logarithmic functions are vertical dilations of each other.

2.12.A.4

The function $f(x) = \ln x$ is a logarithmic function with the natural base e; that is, $\ln x = \log_e x$.

INSTRUCTIONAL PERIODS: 3–4

SKILLS FOCUS

1.A
Solve equations and inequalities represented analytically, with and without technology.

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

1.C
Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

TOPIC 2.13

Exponential and Logarithmic Equations and Inequalities

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.13.A Solve exponential and logarithmic equations and inequalities.	**2.13.A.1** Properties of exponents, properties of logarithms, and the inverse relationship between exponential and logarithmic functions can be used to solve equations and inequalities involving exponents and logarithms. **2.13.A.2** When solving exponential and logarithmic equations found through analytical or graphical methods, the results should be examined for extraneous solutions precluded by the mathematical or contextual limitations. **2.13.A.3** Logarithms can be used to rewrite expressions involving exponential functions in different ways that may reveal helpful information. Specifically, $b^x = c^{(\log_c b)(x)}$.
2.13.B Construct the inverse function for exponential and logarithmic functions.	**2.13.B.1** The function $f(x) = ab^{(x-h)} + k$ is a combination of additive transformations of an exponential function in general form. The inverse of $y = f(x)$ can be found by determining the inverse operations to reverse the mapping. **2.13.B.2** The function $f(x) = a\log_b(x-h) + k$ is a combination of additive transformations of a logarithmic function in general form. The inverse of $y = f(x)$ can be found by determining the inverse operations to reverse the mapping.

TOPIC 2.14

Logarithmic Function Context and Data Modeling

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.B

Apply numerical results in a given mathematical or applied context.

Required Course Content

LEARNING OBJECTIVE

2.14.A

Construct a logarithmic function model.

ESSENTIAL KNOWLEDGE

2.14.A.1

Logarithmic functions are inverses of exponential functions and can be used to model situations involving proportional growth, or repeated multiplication, where the input values change proportionally over equal-length output-value intervals. Alternately, if the output value is a whole number, it indicates how many times the initial value has been multiplied by the proportion.

2.14.A.2

A logarithmic function model can be constructed from an appropriate proportion and a real zero or from two input-output pairs.

2.14.A.3

Logarithmic function models can be constructed by applying transformations to $f(x) = a\log_b x$ based on characteristics of a context or data set.

2.14.A.4

Logarithmic function models can be constructed for a data set with technology using logarithmic regressions.

2.14.A.5

The natural logarithm function is often useful in modeling real-world phenomena.

2.14.A.6

Logarithmic function models can be used to predict values for the dependent variable.

TOPIC 2.15

Semi-log Plots

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.C

Support conclusions or choices with a logical rationale or appropriate data.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
2.15.A Determine if an exponential model is appropriate by examining a semi-log plot of a data set.	**2.15.A.1** In a semi-log plot, one of the axes is logarithmically scaled. When the y-axis of a semi-log plot is logarithmically scaled, data or functions that demonstrate exponential characteristics will appear linear. **2.15.A.2** An advantage of semi-log plots is that a constant never needs to be added to the dependent variable values to reveal that an exponential model is appropriate.
2.15.B Construct the linearization of exponential data.	**2.15.B.1** Techniques used to model linear functions can be applied to a semi-log graph. **2.15.B.2** For an exponential model of the form $y = ab^x$, the corresponding linear model for the semi-log plot is $y = (\log_n b)x + \log_n a$, where $n > 0$ and $n \neq 1$. Specifically, the linear rate of change is $\log_n b$, and the initial linear value is $\log_n a$.

AP PRECALCULUS

UNIT 3

Trigonometric and Polar Functions

AP Precalculus Exam Topics
(required for college calculus placement)

30–35%
AP EXAM WEIGHTING

35–50
CLASS PERIODS

Remember to go to **AP Classroom** to assign students the online **Progress Checks** for this unit.

Whether assigned as homework or completed in class, the **Progress Checks** provide each student with immediate feedback related to this unit's topics and skills.

Progress Check Unit 3 Part 1: Topics 3.1–3.7

Multiple-choice: 21
Free-response: 2

Progress Check Unit 3 Part 2: Topics 3.8–3.15

Multiple-choice: 24
Free-response: 2

UNIT 3

30–35% AP EXAM WEIGHTING | **35–50** CLASS PERIODS

Trigonometric and Polar Functions

ESSENTIAL QUESTIONS

- Since energy usage goes up and down through the year, how can I use trends in data to predict my monthly electricity bills when I get my first apartment?
- How do we model aspects of circular and spinning objects without using complex equations from the *x*-*y* rectangular-based coordinate system?
- How does right triangle trigonometry from geometry relate to trigonometric functions?

←→ Developing Understanding

In Unit 3, students explore trigonometric functions and their relation to the angles and arcs of a circle. Since their output values repeat with every full revolution around the circle, trigonometric functions are ideal for modeling periodic, or repeated pattern phenomena, such as: the highs and lows of a wave, the blood pressure produced by a heart, and the angle from the North Pole to the Sun year to year. Furthermore, periodicity is found in human inventions and social phenomena. For example, moving parts of an analog clock are modeled by a trigonometric function with respect to each other or with respect to time; traffic flow at an intersection over the course of a week demonstrates daily periodicity; and demand for a particular product over the course of a year falls into an annually repeating pattern. Polar functions, which are also explored in this unit, have deep ties to trigonometric functions as they are both based on the circle. Polar functions are defined on the polar coordinate system that uses the circular concepts of radii and angles to describe location instead of rectangular concepts of left-right and up-down, which students have worked with previously. Trigonometry serves as the bridge between the two systems.

Building the Mathematical Practices

1.A 1.B 1.C 2.B 3.A

Students should have multiple experiences transitioning among, and communicating about, the various representations of trigonometric functions, especially sinusoidal functions. It is important that, in addition to solving trigonometric equations and finding equivalent trigonometric expressions, students build sinusoidal models with and without technology and practice constructing different representations. As students transition to thinking in the polar plane, they will refine their communications related to characteristics of functions. The more casual language that students may have adopted such as “goes up” and “goes down” will need to be replaced with more careful language that addresses a function’s behavior related to angles and radii.

Preparing for the AP Exam

As sinusoidal function content pairs well with each of the mathematical practices and skills in the course, students should practice applying each skill to each trigonometric function objective. On the AP Exam, students will need to describe characteristics of sinusoidal functions, such as amplitude, vertical shift, period, and phase shift, give reasons for why chosen values are consistent with information provided, calculate sinusoidal regressions using a graphing calculator, and apply trigonometric function models to contexts. Students should be practicing these skills in daily activities, so they are able to use them fluently on the AP Exam.

UNIT AT A GLANCE

Topic	Instructional Periods	Suggested Skill Focus
3.1 Periodic Phenomena	2	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.2 Sine, Cosine, and Tangent	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.3 Sine and Cosine Function Values	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.
3.4 Sine and Cosine Function Graphs	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.5 Sinusoidal Functions	2–3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.6 Sinusoidal Function Transformations	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
3.7 Sinusoidal Function Context and Data Modeling	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.C** Support conclusions or choices with a logical rationale or appropriate data.
3.8 The Tangent Function	2	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

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UNIT AT A GLANCE *(cont'd)*

Topic	Instructional Periods	Suggested Skill Focus
3.9 Inverse Trigonometric Functions	2–3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
3.10 Trigonometric Equations and Inequalities	3–4	**1.A** Solve equations and inequalities represented analytically, with and without technology. **2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context.
3.11 The Secant, Cosecant, and Cotangent Functions	2	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.12 Equivalent Representations of Trigonometric Functions	3–4	**1.A** Solve equations and inequalities represented analytically, with and without technology. **1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.B** Apply numerical results in a given mathematical or applied context.
3.13 Trigonometry and Polar Coordinates	2–3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.
3.14 Polar Function Graphs	2–3	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
3.15 Rates of Change in Polar Functions	2–3	**3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools. **3.C** Support conclusions or choices with a logical rationale or appropriate data.

Go to **AP Classroom** to assign the **Progress Checks** for Unit 3.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 125 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	3.3	In pairs, students rehearse values of trigonometric functions at multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ using a modified question and answer-type game. Student *A* privately calculates $\sin\left(-\frac{\pi}{3}\right)$ and says, "The answer is $-\frac{\sqrt{3}}{2}$" and Student *B* asks a question such as "What is the sine of [some angle]?" or "What is the cosine of [some angle]? If Student *B* does not guess $\sin\left(-\frac{\pi}{3}\right)$, they work together to determine if Student *B*'s question is also valid for the given answer.
2	3.4	Students are given analytical trigonometric functions paired with graphs on a single sheet of paper or on index cards. Some of the graphs are correct, and others are incorrect. In pairs, students identify the function for which the analytical and graphical representations are consistent and those for which the representations are inconsistent. For graphs that are in error, students construct the appropriate graph.
3	3.6	In pairs, students are given a worksheet containing four functions such as $f(x) = 3\sin(x - \pi) - 4$ and $g(x) = -2\cos(3x) + 1$ along with a list of instructions such as: "Determine the midline," "Determine the period," "Determine the zeros on $[0,\ 2\pi]$," "On what intervals is the function increasing?" The size of the group is equal to the number of questions. Students take turns answering a question of their choice, passing the worksheet between turns. If the receiver of the worksheet disagrees with the passer's answer, they discuss it before the receiver answers a question and returns the worksheet. This continues until all questions are answered for all functions.
4	3.12	Provide each student with the same worksheet containing four trigonometric equations that require using equivalent trigonometric forms (such as sum identity or Pythagorean identity) to solve the equations, such as $2\sin^2 x + \cos x - 1 = 0$. Then, in an activity similar to the one used for Topic 3.6, arrange students in groups of four. Each student solves the first equation and then they pass their papers clockwise to the next student. Each student checks the first equation and, if necessary, discusses any mistakes with the previous student. Each student now solves the second equation on the paper, and the process continues until each student has their original paper back.

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Activity	Topic	Sample Activity
5	**3.14**	In groups of 2 or 3, give students a list of functions in rectangular form such as $f(x) = \sin(2x)$. Students graph the rectangular form on rectangular graph paper then use the information from this graph to sketch the corresponding polar function, $r = f(\theta) = \sin(2\theta)$ on polar graphing paper. Students discuss the relationships between the two graphs, describing intervals of increase and decrease, and intercepts.
6	**3.15**	Students are given a polar function. In pairs, students use their graphing calculator to determine intervals on which the function is positive and increasing, positive and decreasing, negative and increasing, and negative and decreasing. Students compare their answers and verify their results. Students then write, describing how the radius is changing as the angle is changing on each interval, and present this wording to the class.

TOPIC 3.1

Periodic Phenomena

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.1.A Construct graphs of periodic relationships based on verbal representations.	**3.1.A.1** A periodic relationship can be identified between two aspects of a context if, as the input values increase, the output values demonstrate a repeating pattern over successive equal-length intervals. **3.1.A.2** The graph of a periodic relationship can be constructed from the graph of a single cycle of the relationship.
3.1.B Describe key characteristics of a periodic function based on a verbal representation.	**3.1.B.1** The *period* of the function is the smallest positive value k such that $f(x+k)=f(x)$ for all x in the domain. Consequently, the behavior of a periodic function is completely determined by any interval of width k. **3.1.B.2** The period can be estimated by investigating successive equal-length output values and finding where the pattern begins to repeat. **3.1.B.3** Periodic functions take on characteristics of other functions, such as intervals of increase and decrease, different concavities, and various rates of change. However, with periodic functions, all characteristics found in one period of the function will be in every period of the function.

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TOPIC 3.2

Sine, Cosine, and Tangent

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.2.A Determine the sine, cosine, and tangent of an angle using the unit circle.	**3.2.A.1** In the coordinate plane, an angle is in *standard position* when the vertex coincides with the origin and one ray coincides with the positive x-axis. The other ray is called the *terminal ray*. Positive and negative angle measures indicate rotations from the positive x-axis in the counterclockwise and clockwise direction, respectively. Angles in standard position that share a terminal ray differ by an integer number of revolutions. **3.2.A.2** The radian measure of an angle in standard position is the ratio of the length of the arc of a circle centered at the origin subtended by the angle to the radius of that same circle. For a unit circle, which has radius 1, the radian measure is the same as the length of the subtended arc. **3.2.A.3** Given an angle in standard position and a circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The *sine* of the angle is the ratio of the vertical displacement of P from the x-axis to the distance between the origin and point P. Therefore, for a unit circle, the sine of the angle is the y-coordinate of point P.

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LEARNING OBJECTIVE

3.2.A

Determine the sine, cosine, and tangent of an angle using the unit circle.

ESSENTIAL KNOWLEDGE

3.2.A.4

Given an angle in standard position and a circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The *cosine* of the angle is the ratio of the horizontal displacement of P from the y-axis to the distance between the origin and point P. Therefore, for a unit circle, the cosine of the angle is the x-coordinate of point P.

3.2.A.5

Given an angle in standard position, the *tangent* of the angle is the slope, if it exists, of the terminal ray. Because the slope of the terminal ray is the ratio of the vertical displacement to the horizontal displacement over any interval, the tangent of the angle is the ratio of the y-coordinate to the x-coordinate of the point at which the terminal ray intersects the unit circle; alternately, it is the ratio of the angle's sine to its cosine.

TOPIC 3.3

Sine and Cosine Function Values

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.B

Apply numerical results in a given mathematical or applied context.

Required Course Content

LEARNING OBJECTIVE

3.3.A

Determine coordinates of points on a circle centered at the origin.

ESSENTIAL KNOWLEDGE

3.3.A.1

Given an angle of measure θ in standard position and a circle with radius r centered at the origin, there is a point, P, where the terminal ray intersects the circle. The coordinates of point P are $(r\cos\theta, r\sin\theta)$.

3.3.A.2

The geometry of isosceles right and equilateral triangles, while attending to the signs of the values based on the quadrant of the angle, can be used to find exact values for the cosine and sine of angles that are multiples of $\frac{\pi}{4}$ and $\frac{\pi}{6}$ radians and whose terminal rays do not lie on an axis.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 3.4

Sine and Cosine Function Graphs

Required Course Content

LEARNING OBJECTIVE

3.4.A

Construct representations of the sine and cosine functions using the unit circle.

ESSENTIAL KNOWLEDGE

3.4.A.1

Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The sine function, $f(\theta) = \sin\theta$, gives the y-coordinate, or vertical displacement from the x-axis, of point P. The domain of the sine function is all real numbers.

3.4.A.2

As the input values, or angle measures, of the sine function increase, the output values oscillate between -1 and 1, taking every value in between and tracking the vertical distance of points on the unit circle from the x-axis.

3.4.A.3

Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The cosine function, $f(\theta) = \cos\theta$, gives the x-coordinate, or horizontal displacement from the y-axis, of point P. The domain of the cosine function is all real numbers.

3.4.A.4

As the input values, or angle measures, of the cosine function increase, the output values oscillate between -1 and 1, taking every value in between and tracking the horizontal distance of points on the unit circle from the y-axis.

TOPIC 3.5

Sinusoidal Functions

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE

3.5.A

Identify key characteristics of the sine and cosine functions.

ESSENTIAL KNOWLEDGE

3.5.A.1

A *sinusoidal function* is any function that involves additive and multiplicative transformations of $f(\theta) = \sin\theta$. The sine and cosine functions are both sinusoidal functions, with $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$.

3.5.A.2

The period and frequency of a sinusoidal function are reciprocals. The period of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ is 2π, and the frequency is $\frac{1}{2\pi}$.

3.5.A.3

The amplitude of a sinusoidal function is half the difference between its maximum and minimum values. The amplitude of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ is 1.

3.5.A.4

The midline of the graph of a sinusoidal function is determined by the average, or arithmetic mean, of the maximum and minimum values of the function. The midline of the graphs of $y = \sin\theta$ and $y = \cos\theta$ is $y = 0$.

3.5.A.5

As input values increase, the graphs of sinusoidal functions oscillate between concave down and concave up.

3.5.A.6

The graph of $y = \sin\theta$ has rotational symmetry about the origin and is therefore an odd function. The graph of $y = \cos\theta$ has reflective symmetry over the y-axis and is therefore an even function.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

TOPIC 3.6

Sinusoidal Function Transformations

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.6.A Identify the amplitude, vertical shift, period, and phase shift of a sinusoidal function.	**3.6.A.1** Functions that can be written in the form $f(\theta) = a\sin(b(\theta + c)) + d$ or $g(\theta) = a\cos(b(\theta + c)) + d$, where a, b, c, and d are real numbers and $a \neq 0$, are sinusoidal functions and are transformations of the sine and cosine functions. Additive and multiplicative transformations are the same for both sine and cosine because the cosine function is a phase shift of the sine function by $-\frac{\pi}{2}$ units. **3.6.A.2** The graph of the additive transformation $g(\theta) = \sin\theta + d$ of the sine function $f(\theta) = \sin\theta$ is a vertical translation of the graph of f, including its midline, by d units. The same transformation of the cosine function yields the same result. **3.6.A.3** The graph of the additive transformation $g(\theta) = \sin(\theta + c)$ of the sine function $f(\theta) = \sin\theta$ is a horizontal translation, or phase shift, of the graph of f by $-c$ units. The same transformation of the cosine function yields the same result. **3.6.A.4** The graph of the multiplicative transformation $g(\theta) = a\sin\theta$ of the sine function $f(\theta) = \sin\theta$ is a vertical dilation of the graph of f and differs in amplitude by a factor of $\lvert a\rvert$. The same transformation of the cosine function yields the same result.

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LEARNING OBJECTIVE

3.6.A

Identify the amplitude, vertical shift, period, and phase shift of a sinusoidal function.

ESSENTIAL KNOWLEDGE

3.6.A.5

The graph of the multiplicative transformation $g(\theta) = \sin(b\theta)$ of the sine function $f(\theta) = \sin\theta$ is a horizontal dilation of the graph of f and differs in period by a factor of $\left|\frac{1}{b}\right|$. The same transformation of the cosine function yields the same result.

3.6.A.6

The graph of $y = f(\theta) = a\sin(b(\theta + c)) + d$ has an amplitude of $|a|$ units, a period of $\left|\frac{1}{b}\right|2\pi$ units, a midline vertical shift of d units from $y = 0$, and a phase shift of $-c$ units. The same transformations of the cosine function yield the same results.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C
Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.C
Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 3.7

Sinusoidal Function Context and Data Modeling

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.7.A Construct sinusoidal function models of periodic phenomena.	**3.7.A.1** The smallest interval of input values over which the maximum or minimum output values start to repeat, that is, the input-value interval between consecutive maxima or consecutive minima, can be used to determine or estimate the period and frequency for a sinusoidal function model. **3.7.A.2** The maximum and minimum output values can be used to determine or estimate the amplitude and vertical shift for a sinusoidal function model. **3.7.A.3** An actual pair of input-output values can be compared to pairs of input-output values produced by a sinusoidal function model to determine or estimate a phase shift for the model. **3.7.A.4.** Sinusoidal function models can be constructed for a data set with technology by estimating key values or using sinusoidal regressions. **3.7.A.5** Sinusoidal functions that model a data set are frequently only useful over their contextual domain and can be used to predict values of the dependent variable from values of the independent variable.

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TOPIC 3.8
The Tangent Function

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.8.A Construct representations of the tangent function using the unit circle.	**3.8.A.1** Given an angle of measure θ in standard position and a unit circle centered at the origin, there is a point, P, where the terminal ray intersects the circle. The tangent function, $f(\theta) = \tan\theta$, gives the slope of the terminal ray. **3.8.A.2** Because the slope of the terminal ray is the ratio of the change in the y-values to the change in the x-values between any two points on the ray, the tangent function is also the ratio of the sine function to the cosine function. Therefore, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, where $\cos\theta \neq 0$.
3.8.B Describe key characteristics of the tangent function.	**3.8.B.1** Because the slope values of the terminal ray repeat every one-half revolution of the circle, the tangent function has a period of π. **3.8.B.2** The tangent function demonstrates periodic asymptotic behavior at input values $\theta = \frac{\pi}{2} + k\pi$, for integer values of k, because $\cos\theta = 0$ at those values. **3.8.B.3** The tangent function increases and its graph changes from concave down to concave up between consecutive asymptotes.

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LEARNING OBJECTIVE

3.8.C

Describe additive and multiplicative transformations involving the tangent function.

ESSENTIAL KNOWLEDGE

3.8.C.1

The graph of the additive transformation $g(\theta) = \tan\theta + d$ of the tangent function $f(\theta) = \tan\theta$ is a vertical translation of the graph of f and the line containing its points of inflection by d units.

3.8.C.2

The graph of the additive transformation $g(\theta) = \tan(\theta + c)$ of the tangent function $f(\theta) = \tan\theta$ is a horizontal translation, or phase shift, of the graph of f by $-c$ units.

3.8.C.3

The graph of the multiplicative transformation $g(\theta) = a\tan\theta$ of the tangent function $f(\theta) = \tan\theta$ is a vertical dilation of the graph of f by a factor of $|a|$. If $a < 0$, the transformation involves a reflection over the x-axis.

3.8.C.4

The graph of the multiplicative transformation $g(\theta) = \tan(b\theta)$ of the tangent function $f(\theta) = \tan\theta$ is a horizontal dilation of the graph of f and differs in period by a factor of $\left|\frac{1}{b}\right|$. If $b < 0$, the transformation involves a reflection over the y-axis.

3.8.C.5

The graph of $y = f(\theta) = a\tan(b(\theta + c)) + d$ is a vertical dilation of the graph of $y = \tan\theta$ by a factor of $|a|$, has a period of $\left|\frac{1}{b}\right|\pi$ units, is a vertical shift of the line containing the points of inflection of the graph of $y = \tan\theta$ by d units, and is a phase shift of $-c$ units.

TOPIC 3.9

Inverse Trigonometric Functions

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

Required Course Content

LEARNING OBJECTIVE

3.9.A

Construct analytical and graphical representations of the inverse of the sine, cosine, and tangent functions over a restricted domain.

ESSENTIAL KNOWLEDGE

3.9.A.1

For inverse trigonometric functions, the input and output values are switched from their corresponding trigonometric functions, so the output value of an inverse trigonometric function is often interpreted as an angle measure and the input is a value in the range of the corresponding trigonometric function.

3.9.A.2

The inverse trigonometric functions are called *arcsine*, *arccosine*, and *arctangent* (also represented as $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$). Because the corresponding trigonometric functions are periodic, they are only invertible if they have restricted domains.

3.9.A.3

In order to define their respective inverse functions, the domain of the sine function is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the cosine function to $[0, \pi]$, and the tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

INSTRUCTIONAL PERIODS: 3–4

SKILLS FOCUS

1.A
Solve equations and inequalities represented analytically, with and without technology.

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.B
Apply numerical results in a given mathematical or applied context.

TOPIC 3.10

Trigonometric Equations and Inequalities

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.10.A Solve equations and inequalities involving trigonometric functions.	**3.10.A.1** Inverse trigonometric functions are useful in solving equations and inequalities involving trigonometric functions, but solutions may need to be modified due to domain restrictions. **3.10.A.2** Because trigonometric functions are periodic, there are often infinitely many solutions to trigonometric equations. **3.10.A.3** In trigonometric equations and inequalities arising from a contextual scenario, there is often a domain restriction that can be implied from the context, which limits the number of solutions.

TOPIC 3.11

The Secant, Cosecant, and Cotangent Functions

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.B
Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.11.A Identify key characteristics of functions that involve quotients of the sine and cosine functions.	**3.11.A.1** The secant function, $f(\theta) = \sec\theta$, is the reciprocal of the cosine function, where $\cos\theta \neq 0$. **3.11.A.2** The cosecant function, $f(\theta) = \csc\theta$, is the reciprocal of the sine function, where $\sin\theta \neq 0$. **3.11.A.3** The graphs of the secant and cosecant functions have vertical asymptotes where cosine and sine are zero, respectively, and have a range of $(-\infty, -1] \cup [1, \infty)$. **3.11.A.4** The cotangent function, $f(\theta) = \cot\theta$, is the reciprocal of the tangent function, where $\tan\theta \neq 0$. Equivalently, $\cot\theta = \frac{\cos\theta}{\sin\theta}$, where $\sin\theta \neq 0$. **3.11.A.5** The graph of the cotangent function has vertical asymptotes for domain values where $\tan\theta = 0$ and is decreasing between consecutive asymptotes.

TOPIC 3.12

Equivalent Representations of Trigonometric Functions

INSTRUCTIONAL PERIODS: 3–4

SKILLS FOCUS

1.A
Solve equations and inequalities represented analytically, with and without technology.

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.B
Apply numerical results in a given mathematical or applied context.

Required Course Content

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
3.12.A Rewrite trigonometric expressions in equivalent forms with the Pythagorean identity.	**3.12.A.1** The Pythagorean Theorem can be applied to right triangles with points on the unit circle at coordinates $(\cos\theta, \sin\theta)$, resulting in the Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$. **3.12.A.2** The Pythagorean identity can be algebraically manipulated into other forms involving trigonometric functions, such as $\tan^2\theta = \sec^2\theta - 1$, and can be used to establish other trigonometric relationships, such as $\arcsin x = \arccos\left(\sqrt{1-x^2}\right)$, with appropriate domain restrictions.
3.12.B Rewrite trigonometric expressions in equivalent forms with sine and cosine sum identities.	**3.12.B.1** The sum identity for sine is $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. **3.12.B.2** The sum identity for cosine is $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. **3.12.B.3** The sum identities for sine and cosine can also be used as difference and double-angle identities. **3.12.B.4** Properties of trigonometric functions, known trigonometric identities, and other algebraic properties can be used to verify additional trigonometric identities.

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LEARNING OBJECTIVE

3.12.C

Solve equations using equivalent analytic representations of trigonometric functions.

ESSENTIAL KNOWLEDGE

3.12.C.1

A specific equivalent form involving trigonometric expressions can make information more accessible.

3.12.C.2

Equivalent trigonometric forms may be useful in solving trigonometric equations and inequalities.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

TOPIC 3.13

Trigonometry and Polar Coordinates

Required Course Content

LEARNING OBJECTIVE

3.13.A

Determine the location of a point in the plane using both rectangular and polar coordinates.

ESSENTIAL KNOWLEDGE

3.13.A.1

The polar coordinate system is based on a grid of circles centered at the origin and on lines through the origin. Polar coordinates are defined as an ordered pair, (r, θ), such that $|r|$ represents the radius of the circle on which the point lies, and θ represents the measure of an angle in standard position whose terminal ray includes the point. In the polar coordinate system, the same point can be represented many ways.

3.13.A.2

The coordinates of a point in the polar coordinate system, (r, θ), can be converted to coordinates in the rectangular coordinate system, (x, y), using $x = r\cos\theta$ and $y = r\sin\theta$.

3.13.A.3

The coordinates of a point in the rectangular coordinate system, (x, y), can be converted to coordinates in the polar coordinate system, (r, θ), using $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$ for $x > 0$ or $\theta = \arctan\left(\frac{y}{x}\right) + \pi$ for $x < 0$.

3.13.A.4

A complex number can be understood as a point in the complex plane and can be determined by its corresponding rectangular or polar coordinates. When the complex number has the rectangular coordinates $(a,\ b)$, it can be expressed as $a + bi$. When the complex number has polar coordinates $(r,\ \theta)$, it can be expressed as $(r\cos\theta) + i(r\sin\theta)$.

TOPIC 3.14

Polar Function Graphs

Required Course Content

LEARNING OBJECTIVE

3.14.A

Construct graphs of polar functions.

ESSENTIAL KNOWLEDGE

3.14.A.1

The graph of the function $r = f(\theta)$ in polar coordinates consists of input-output pairs of values where the input values are angle measures and the output values are radii.

3.14.A.2

The domain of the polar function $r = f(\theta)$, given graphically, can be restricted to a desired portion of the function by selecting endpoints corresponding to the desired angle and radius.

3.14.A.3

When graphing polar functions in the form of $r = f(\theta)$, changes in input values correspond to changes in angle measure from the positive x-axis, and changes in output values correspond to changes in distance from the origin.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

INSTRUCTIONAL PERIODS: 2–3

SKILLS FOCUS

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

3.C

Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 3.15

Rates of Change in Polar Functions

Required Course Content

LEARNING OBJECTIVE

3.15.A

Describe characteristics of the graph of a polar function.

ESSENTIAL KNOWLEDGE

3.15.A.1

If a polar function, $r = f(\theta)$, is positive and increasing or negative and decreasing, then the distance between $f(\theta)$ and the origin is increasing.

3.15.A.2

If a polar function, $r = f(\theta)$, is positive and decreasing or negative and increasing, then the distance between $f(\theta)$ and the origin is decreasing.

3.15.A.3

For a polar function, $r = f(\theta)$, if the function changes from increasing to decreasing or decreasing to increasing on an interval, then the function has a relative extremum on the interval corresponding to a point relatively closest to or farthest from the origin.

3.15.A.4

The average rate of change of r with respect to θ over an interval of θ is the ratio of the change in the radius values to the change in θ over an interval of θ. Graphically, the average rate of change indicates the rate at which the radius is changing per radian.

3.15.A.5

The average rate of change of r with respect to θ over an interval of θ can be used to estimate values of the function within the interval.

AP PRECALCULUS

UNIT 4

Functions Involving Parameters, Vectors, and Matrices

Additional Topics Available to Schools
(not included on AP Precalculus Exam)

0%
AP EXAM WEIGHTING

35
CLASS PERIODS

Remember to go to **AP Classroom** to assign students the online optional **Progress Checks** for this unit.

Whether assigned as homework or completed in class, the **Progress Checks** provide each student with immediate feedback related to this unit's topics and skills.

Progress Check Unit 4 Part 1: Topics 4.1–4.7

Multiple-choice: 24
Free-response: 2

Progress Check Unit 4 Part 2: Topics 4.8–4.14

Multiple-choice: 21
Free-response: 2

UNIT
4

0% AP EXAM WEIGHTING

35 CLASS PERIODS

Functions Involving Parameters, Vectors, and Matrices

ESSENTIAL QUESTIONS

- How can we determine when the populations of species in an ecosystem will be relatively steady?
- How can we analyze the vertical and horizontal aspects of motion independently?
- How does high resolution computer-generated imaging achieve smooth and realistic motion on screen with so many pixels?

Developing Understanding

In Unit 4, students explore function types that expand their understanding of the function concept. Parametric functions have multiple dependent variables' values paired with a single input variable or parameter. Modeling scenarios with parametric functions allows students to explore change in terms of components. This component-based understanding is important not only in calculus but in all fields of the natural and social sciences where we seek to understand one aspect of a phenomenon independent of other confounding aspects. Another major function type in this unit involves matrices mapping a set of input vectors to output vectors. The capacity to map large quantities of vectors instantaneously is the basis for vector-based computer graphics. While students may see their favorite video game character trip and fall or seemingly move closer or farther, matrices implement a rotation on a set of vectors or a dilation on a set of vectors. The power of matrices to map vectors is not limited to graphics but to any system that can be expressed in terms of components of vectors such as electrical systems, network connections, and regional population distribution changes over time. Vectors and matrices are also powerful tools of data science as they can be used to model aspects of complex scientific and social science phenomena.

Building the Mathematical Practices

2.A 2.B 3.A 3.C

When encountering new function types, students should engage with multiple representations of each function type and practice communicating precise characteristics of these function types. For parametric and vector-valued functions, students will need to use care in communicating about the position or velocity of an object, depending on the function that is given. Students should practice the precise language used with particle motion in the plane and refer specifically to position, direction, and motion. It will be valuable for students to provide clear rationales when setting up and working with matrices as linear transformation functions on vectors. Students should explain why they took the steps they did.

Preparing for the AP Exam

Unit 4 topics are excluded from the AP Exam. The AP Exam assesses topics in Unit 1, 2, and 3 as these topics are required for college credit and/or placement. When teachers and schools choose to include topics in Unit 4, students will use technology in ways that are new and unfamiliar. Students should practice setting up appropriate viewing windows and parameter restrictions when graphing parametric functions. Students should practice building matrices, manipulating matrices, and calculating the inverse of a matrix, where defined, with graphing calculators. Topic Questions and Progress Checks found in AP Classroom provide opportunities to practice with technology. Many of these practices will not only help students learn topics in this unit, but solidify understandings of topics in Units 1, 2, and 3.

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UNIT AT A GLANCE

Topic	Instructional Periods	Suggested Skill Focus
4.1 Parametric Functions	2	**1.A** Solve equations and inequalities represented analytically, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
4.2 Parametric Functions Modeling Planar Motion	2	**3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools. **3.B** Apply numerical results in a given mathematical or applied context.
4.3 Parametric Functions and Rates of Change	2	**3.B** Apply numerical results in a given mathematical or applied context. **3.C** Support conclusions or choices with a logical rationale or appropriate data.
4.4 Parametrically Defined Circles and Lines	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.
4.5 Implicitly Defined Functions	2	**2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
4.6 Conic Sections	3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **2.B** Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.
4.7 Parametrization of Implicitly Defined Functions	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

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UNIT AT A GLANCE *(cont'd)*

Topic	Instructional Periods	Suggested Skill Focus
4.8 Vectors	3	**2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools. **3.B** Apply numerical results in a given mathematical or applied context.
4.9 Vector-Valued Functions	1	**3.C** Support conclusions or choices with a logical rationale or appropriate data.
4.10 Matrices	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.B** Apply numerical results in a given mathematical or applied context.
4.11 The Inverse and Determinant of a Matrix	2	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **3.B** Apply numerical results in a given mathematical or applied context.
4.12 Linear Transformations and Matrices	1	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.
4.13 Matrices as Functions	3	**1.B** Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context. **2.A** Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology. **3.A** Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.
4.14 Matrices Modeling Contexts	3	**1.C** Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology. **3.B** Apply numerical results in a given mathematical or applied context. **3.C** Support conclusions or choices with a logical rationale or appropriate data.

Go to **AP Classroom** to assign the optional **Progress Checks** for Unit 4.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 125 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	**4.1**	In pairs, students discuss why the graphs of parametric functions can fail the "vertical line test" (and yet they are functions), but the graphs of many other previously studied function types must pass the vertical line test.
2	**4.2**	Students are put into groups of 3-4 and are given a paper with a parametric function used to model planar motion. Each group gets a different function. Each person in the group writes a question on the paper that can be answered based on the given parametric function. On a separate blank sheet of paper, the group members verify that each question is answerable. Then, each group trades their question paper with another group. Each group works together to answer the questions on the received paper.
3	**4.6**	Students are provided with four graphs: a graph of a parabola that opens up or opens down with the vertex at the origin, a circle centered at the origin, an ellipse not centered at the origin, and a hyperbola with center at the origin. In groups, students develop a reason for why each of the four graphs does not belong to the set of four. (For example: The parabola is the only one that can be explicitly defined with y as a function of x; the circle is the only one with infinitely many lines of symmetry; the ellipse does not belong because it is not symmetric to the y-axis; the hyperbola does not belong because it has two disconnected pieces.) This type of activity can be repeated with four different graphs of a parabola, a circle, an ellipse, and a hyperbola, where students come up with the graphs.
4	**4.8**	Using information about the definition of the dot product, students construct an argument to explain why two nonzero vectors are perpendicular if, and only if, their dot product is equal to zero.
5	**4.12**	In pairs, students create a graphic organizer to highlight the similarities and differences between linear functions and linear transformations.
6	**4.14**	Students are given scenarios involving transitions between two states that can be modeled by matrices and linear transformations, such as the number of students who choose between two lunch options each day. For each model, in pairs, students explore past and future states for 2 steps using technology. Based on these limited data, each student tries to predict the steady state. Students then examine their hypotheses by calculating future states for 10, 100, and 1000 steps.

TOPIC 4.1

Parametric Functions

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.A
Solve equations and inequalities represented analytically, with and without technology.

2.B
Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.1.A
Construct a graph or table of values for a parametric function represented analytically.

ESSENTIAL KNOWLEDGE

4.1.A.1
A parametric function in $\mathbb{R}^2$, the set of all ordered pairs of two real numbers, consists of a set of two parametric equations in which two dependent variables, x and y, are dependent on a single independent variable, t, called the *parameter*.

4.1.A.2
Because variables x and y are dependent on the independent variable, t, the coordinates (x_i, y_i) at time t_i can be written as functions of t and can be expressed as the single parametric function $f(t) = (x(t), y(t))$, where in this case x and y are names of two functions.

4.1.A.3
A numerical table of values can be generated for the parametric function $f(t) = (x(t), y(t))$ by evaluating x_i and y_i at several values of t_i within the domain.

4.1.A.4
A graph of a parametric function can be sketched by connecting several points from the numerical table of values in order of increasing value of t.

4.1.A.5
The domain of the parametric function f is often restricted, which results in start and end points on the graph of f.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

3.B
Apply numerical results in a given mathematical or applied context.

TOPIC 4.2

Parametric Functions Modeling Planar Motion

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.2.A
Identify key characteristics of a parametric planar motion function that are related to position.

ESSENTIAL KNOWLEDGE

4.2.A.1
A parametric function given by $f(t) = (x(t), y(t))$ can be used to model particle motion in the plane. The graph of this function indicates the position of a particle at time t.

4.2.A.2
The horizontal and vertical extrema of a particle's motion can be determined by identifying the maximum and minimum values of the functions $x(t)$ and $y(t)$, respectively.

4.2.A.3
The real zeros of the function $x(t)$ correspond to y-intercepts, and the real zeros of $y(t)$ correspond to x-intercepts.

TOPIC 4.3

Parametric Functions and Rates of Change

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

3.B

Apply numerical results in a given mathematical or applied context.

3.C

Support conclusions or choices with a logical rationale or appropriate data.

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.3.A

Identify key characteristics of a parametric planar motion function that are related to direction and rate of change.

ESSENTIAL KNOWLEDGE

4.3.A.1

As the parameter increases, the direction of planar motion of a particle can be analyzed in terms of x and y independently. If $x(t)$ is increasing or decreasing, the direction of motion is to the right or left, respectively. If $y(t)$ is increasing or decreasing, the direction of motion is up or down, respectively.

4.3.A.2

At any given point in the plane, the direction of planar motion may be different for different values of t.

4.3.A.3

The same curve in the plane can be parametrized in different ways and can be traversed in different directions with different parametric functions.

4.3.A.4

Over a given interval $[t_1, t_2]$ within the domain, the average rate of change can be computed for $x(t)$ and $y(t)$ independently. The ratio of the average rate of change of y to the average rate of change of x gives the slope of the graph between the points on the curve corresponding to t_1 and t_2, so long as the average rate of change of $x(t) \neq 0$.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

1.C
Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

TOPIC 4.4

Parametrically Defined Circles and Lines

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.4.A
Express motion around a circle or along a line segment parametrically.

ESSENTIAL KNOWLEDGE

4.4.A.1
A complete counterclockwise revolution around the unit circle that starts and ends at $(1,\ 0)$ and is centered at the origin can be modeled by $(x(t),\ y(t)) = (\cos t,\ \sin t)$ with domain $0 \leq t \leq 2\pi$.

4.4.A.2
Transformations of the parametric function $(x(t),\ y(t)) = (\cos t,\ \sin t)$ can model any circular path traversed in the plane.

4.4.A.3
A linear path along the line segment from the point $(x_1,\ y_1)$ to the point $(x_2,\ y_2)$ can be parametrized many ways, including using an initial position $(x_1,\ y_1)$ and rates of change for x with respect to t and y with respect to t.

TOPIC 4.5

Implicitly Defined Functions

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.5.A Construct a graph of an equation involving two variables.	**4.5.A.1** An equation involving two variables can implicitly describe one or more functions. **4.5.A.2** An equation involving two variables can be graphed by finding solutions to the equation. **4.5.A.3** Solving for one of the variables in an equation involving two variables can define a function whose graph is part or all of the graph of the equation.
4.5.B Determine how the two quantities related in an implicitly defined function vary together.	**4.5.B.1** For ordered pairs on the graph of an implicitly defined function that are close together, if the ratio of the change in the two variables is positive, then the two variables simultaneously increase or both decrease; conversely, if the ratio is negative, then as one variable increases, the other decreases. **4.5.B.2** The rate of change of x with respect to y or of y with respect to x can be zero, indicating vertical or horizontal intervals, respectively.

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

2.B

Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

3.A

Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

INSTRUCTIONAL PERIODS: 3

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

2.B
Construct equivalent graphical, numerical, analytical, and verbal representations of functions that are useful in a given mathematical or applied context, with and without technology.

TOPIC 4.6
Conic Sections

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.6.A
Represent conic sections with horizontal or vertical symmetry analytically.

ESSENTIAL KNOWLEDGE

4.6.A.1
A parabola with vertex (h, k) can, if $a \neq 0$, be represented analytically as $x - h = a(y - k)^2$ if it opens left or right, or as $y - k = a(x - h)^2$ if it opens up or down.

4.6.A.2
An ellipse centered at (h, k) with horizontal radius a and vertical radius b can be represented analytically as $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. A circle is a special case of an ellipse where $a = b$.

4.6.A.3
A hyperbola centered at (h, k) with vertical and horizonal lines of symmetry can be represented algebraically as $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ for a hyperbola opening left and right, or as $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ for a hyperbola opening up and down. The asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.

TOPIC 4.7

Parametrization of Implicitly Defined Functions

Additional Topic Available to Schools

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

2.A

Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.7.A Represent a curve in the plane parametrically.	**4.7.A.1** A parametrization $(x(t), y(t))$ for an implicitly defined function will, when $x(t)$ and $y(t)$ are substituted for x and y, respectively, satisfy the corresponding equation for every value of t in the domain. **4.7.A.2** If f is a function of x, then $y = f(x)$ can be parametrized as $(x(t), y(t)) = (t, f(t))$. If f is invertible, its inverse can be parametrized as $(x(t), y(t)) = (f(t), t)$ for an appropriate interval of t.
4.7.B Represent conic sections parametrically.	**4.7.B.1** A parabola can be parametrized in the same way that any equation that can be solved for x or y can be parametrized. Equations that can be solved for x can be parametrized as $(x(t), y(t)) = (f(t), t)$ by solving for x and replacing y with t. Equations that can be solved for y can be parametrized as $(x(t), y(t)) = (t, f(t))$ by solving for y and replacing x with t. **4.7.B.2** An ellipse can be parametrized using the trigonometric functions $x(t) = h + a\cos t$ and $y(t) = k + b\sin t$ for $0 \le t \le 2\pi$. **4.7.B.3** A hyperbola can be parametrized using trigonometric functions. For a hyperbola that opens left and right, the functions are $x(t) = h + a\sec t$ and $y(t) = k + b\tan t$ for $0 \le t \le 2\pi$. For a hyperbola that opens up and down, the functions are $x(t) = h + a\tan t$ and $y(t) = k + b\sec t$ for $0 \le t \le 2\pi$.

INSTRUCTIONAL PERIODS: 3

SKILLS FOCUS

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

3.B
Apply numerical results in a given mathematical or applied context.

TOPIC 4.8
Vectors

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.8.A Identify characteristics of a vector.	**4.8.A.1** A vector is a directed line segment. When a vector is placed in the plane, the point at the beginning of the line segment is called the *tail*, and the point at the end of the line segment is called the *head*. The length of the line segment is the *magnitude* of the vector. **4.8.A.2** A vector $\overrightarrow{P_1P_2}$ with two components can be plotted in the xy-plane from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$. The vector is identified by a and b, where $a = x_2 - x_1$ and $b = y_2 - y_1$. The vector can be expressed as $\langle a, b \rangle$. A zero vector $\langle 0, 0 \rangle$ is the trivial case when $P_1 = P_2$. **4.8.A.3** The *direction* of the vector is parallel to the line segment from the origin to the point with coordinates (a, b). The magnitude of the vector is the square root of the sum of the squares of the components. **4.8.A.4** For a vector represented geometrically in the plane, the components of the vector can be found using trigonometry.
4.8.B Determine sums and products involving vectors.	**4.8.B.1** The multiplication of a constant and a vector results in a new vector whose components are found by multiplying the constant by each of the components of the original vector. The new vector is parallel to the original vector.

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LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.8.B Determine sums and products involving vectors.	**4.8.B.2** The sum of two vectors in $\mathbb{R}^2$ is a new vector whose components are found by adding the corresponding components of the original vectors. The new vector can be represented graphically as a vector whose tail corresponds to the tail of the first vector and whose head corresponds to the head of the second vector when the second vector's tail is located at the first vector's head. **4.8.B.3** The dot product of two vectors is the sum of the products of their corresponding components. That is, $\langle a_1, b_1 \rangle \cdot \langle a_2, b_2 \rangle = a_1a_2 + b_1b_2$.
4.8.C Determine a unit vector for a given vector.	**4.8.C.1** A *unit vector* is a vector of magnitude 1. A unit vector in the same direction as a given nonzero vector can be found by scalar multiplying the vector by the reciprocal of its magnitude. **4.8.C.2** The vector $\langle a, b \rangle$ can be expressed as $a\vec{i} + b\vec{j}$ in $\mathbb{R}^2$, where $\vec{i}$ and $\vec{j}$ are unit vectors in the x and y directions, respectively. That is, $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$.
4.8.D Determine angle measures between vectors and magnitudes of vectors involved in vector addition.	**4.8.D.1** The dot product is geometrically equivalent to the product of the magnitudes of the two vectors and the cosine of the angle between them. Therefore, if the dot product of two nonzero vectors is zero, then the vectors are perpendicular. **4.8.D.2** The Law of Sines and Law of Cosines can be used to determine side lengths and angle measures of triangles formed by vector addition.

INSTRUCTIONAL PERIODS: 1

SKILLS FOCUS

3.C

Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 4.9

Vector-Valued Functions

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.9.A

Represent planar motion in terms of vector-valued functions.

ESSENTIAL KNOWLEDGE

4.9.A.1

The position of a particle moving in a plane that is given by the parametric function $f(t) = (x(t), y(t))$ may be expressed as a *vector-valued function*, $p(t) = x(t)\vec{i} + y(t)\vec{j}$ or $p(t) = \langle x(t), y(t) \rangle$. The magnitude of the position vector at time t gives the distance of the particle from the origin.

4.9.A.2

The vector-valued function $v(t) = \langle x(t), y(t) \rangle$ can be used to express the velocity of a particle moving in a plane at different times, t. At time t, the sign of $x(t)$ indicates if the particle is moving right or left, and the sign of $y(t)$ indicates if the particle is moving up or down. The magnitude of the velocity vector at time t gives the speed of the particle.

TOPIC 4.10

Matrices

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.B

Apply numerical results in a given mathematical or applied context.

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.10.A

Determine the product of two matrices.

ESSENTIAL KNOWLEDGE

4.10.A.1

An $n \times m$ matrix is an array consisting of n rows and m columns.

4.10.A.2

Two matrices can be multiplied if the number of columns in the first matrix equals the number of rows in the second matrix. The product of the matrices is a new matrix in which the component in the ith row and jth column is the dot product of the ith row of the first matrix and the jth column of the second matrix.

TOPIC 4.11

The Inverse and Determinant of a Matrix

INSTRUCTIONAL PERIODS: 2

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

3.B
Apply numerical results in a given mathematical or applied context.

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.11.A Determine the inverse of a 2×2 matrix.	**4.11.A.1** The identity matrix, I, is a square matrix consisting of 1s on the diagonal from the top left to bottom right and 0s everywhere else. **4.11.A.2** Multiplying a square matrix by its corresponding identity matrix results in the original square matrix. **4.11.A.3** The product of a square matrix and its inverse, when it exists, is the identity matrix of the same size. **4.11.A.4** The inverse of a 2×2 matrix, when it exists, can be calculated with or without technology.
4.11.B Apply the value of the determinant to invertibility and vectors.	**4.11.B.1** The determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. The determinant can be calculated with or without technology and is denoted $\det(A)$. **4.11.B.2** If a 2×2 matrix consists of two column or row vectors from $\mathbb{R}^2$, then the nonzero absolute value of the determinant of the matrix is the area of the parallelogram spanned by the vectors represented in the columns or rows of the matrix. If the determinant equals 0, then the vectors are parallel. **4.11.B.3** The square matrix A has an inverse if and only if $\det(A) \neq 0$.

TOPIC 4.12

Linear Transformations and Matrices

INSTRUCTIONAL PERIODS: 1

SKILLS FOCUS

1.B

Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

Additional Topic Available to Schools

LEARNING OBJECTIVE

4.12.A

Determine the output vectors of a linear transformation using a 2×2 matrix.

ESSENTIAL KNOWLEDGE

4.12.A.1

A *linear transformation* is a function that maps an input vector to an output vector such that each component of the output vector is the sum of constant multiples of the input vector components.

4.12.A.2

A linear transformation will map the zero vector to the zero vector.

4.12.A.3

A single vector in $\mathbb{R}^2$ can be expressed as a 2×1 matrix. A set of n vectors in $\mathbb{R}^2$ can be expressed as a $2 \times n$ matrix.

4.12.A.4

For a linear transformation, L, from $\mathbb{R}^2$ to $\mathbb{R}^2$, there is a unique 2×2 matrix, A, such that $L(\vec{v}) = A\vec{v}$ for vectors in $\mathbb{R}^2$. Conversely, for a given 2×2 matrix, A, the function $L(\vec{v}) = A\vec{v}$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

4.12.A.5

Multiplication of a 2×2 transformation matrix, A, and a $2 \times n$ matrix of n input vectors gives a $2 \times n$ matrix of the n output vectors for the linear transformation $L(\vec{v}) = A\vec{v}$.

INSTRUCTIONAL PERIODS: 3

SKILLS FOCUS

1.B
Express functions, equations, or expressions in analytically equivalent forms that are useful in a given mathematical or applied context.

2.A
Identify information from graphical, numerical, analytical, and verbal representations to answer a question or construct a model, with and without technology.

3.A
Describe the characteristics of a function with varying levels of precision, depending on the function representation and available mathematical tools.

TOPIC 4.13

Matrices as Functions

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.13.A Determine the association between a linear transformation and a matrix.	**4.13.A.1** The linear transformation mapping $\langle x, y \rangle$ to $\langle a_{11}x + a_{12}y, a_{21}x + a_{22}y \rangle$ is associated with the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. **4.13.A.2** The mapping of the unit vectors in a linear transformation provides valuable information for determining the associated matrix. **4.13.A.3** The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is associated with a linear transformation of vectors that rotates every vector an angle θ counterclockwise about the origin. **4.13.A.4** The absolute value of the determinant of a 2×2 transformation matrix gives the magnitude of the dilation of regions in $\mathbb{R}^2$ under the transformation.
4.13.B Determine the composition of two linear transformations.	**4.13.B.1** The composition of two linear transformations is a linear transformation. **4.13.B.2** The matrix associated with the composition of two linear transformations is the product of the matrices associated with each linear transformation.

continued on next page

LEARNING OBJECTIVE

4.13.C

Determine the inverse of a linear transformation.

ESSENTIAL KNOWLEDGE

4.13.C.1

Two linear transformations are inverses if their composition maps any vector to itself.

4.13.C.2

If a linear transformation, L, is given by $L(\vec{v}) = A\vec{v}$, then its inverse transformation is given by $L^{-1}(\vec{v}) = A^{-1}\vec{v}$, where A^{-1} is the inverse of the matrix A.

INSTRUCTIONAL PERIODS: 3

SKILLS FOCUS

1.C

Construct new functions, using transformations, compositions, inverses, or regressions, that may be useful in modeling contexts, criteria, or data, with and without technology.

3.B

Apply numerical results in a given mathematical or applied context.

3.C

Support conclusions or choices with a logical rationale or appropriate data.

TOPIC 4.14

Matrices Modeling Contexts

Additional Topic Available to Schools

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
4.14.A Construct a model of a scenario involving transitions between two states using matrices.	**4.14.A.1** A contextual scenario can indicate the rate of transitions between states as percent changes. A matrix can be constructed based on these rates to model how states change over discrete intervals.
4.14.B Apply matrix models to predict future and past states for n transition steps.	**4.14.B.1** The product of a matrix that models transitions between states and a corresponding state vector can predict future states. **4.14.B.2** Repeated multiplication of a matrix that models the transitions between states and corresponding resultant state vectors can predict the steady state, a distribution between states that does not change from one step to the next. **4.14.B.3** The product of the inverse of a matrix that models transitions between states and a corresponding state vector can predict past states.